MA4002 Final Exam Answers, Spring 2016

1.(a) Velocity: $v(t) = 5 + \int_0^t \frac{1}{\sqrt{s+1}} ds = 3 + 2\sqrt{t+1}$.

Distance $s(T) = \int_0^T v(t)dt = 3\,T + \frac{4}{3}[(T+1)^{3/2} - 1]$m and $s(3) = \frac{55}{3} \approx \boxed{18.3333\,\text{m}}$.

(b) Intercepts: $x = 1, 2$. Using cylindrical shells:

$V = \int_1^2 2\pi x\,[(4x - x^2 - 2) - x]\,dx = \int_1^2 \pi(2x^3 - \frac{1}{2}x^4 - 2x^2)\,dx = \frac{\pi}{2} \approx 1.570796327$.

(c) Integrating by parts using $u = x^n$ and $dv = e^{x/2}\,dx$ yields the reduction formula

$I_n = \int_0^2 x^n\, e^{x/2}\,dx = 2x^n\, e^{x/2}|_0^2 - 2n \cdot I_{n-1} = \boxed{2^{n+1}e - 2n \cdot I_{n-1}}$. Next, $I_0 = 2e - 2 \approx 3.436563656$

implies $I_1 = 2^2 e - 2 \cdot [2e-2] = 4$, and $I_2 = 2^3 e - 4 \cdot [4] = 8e - 16 \approx 5.74625462$.

(d) $f_x = (x + y^2)\,\cos x + \sin x,\ f_y = 2y\,\sin x$,

$f_{xx} = -(x+y^2)\,\sin x + 2\cos x,\ f_{yy} = 2\sin x,\ f_{xy} = 2y\,\cos x$.

(e) $x_n = 0.2n$. Start with $y_0 = 1$. $y_{n+1} = y_n + \frac{1}{2}\,0.2\,[\exp(x_n^3 - y_n) + \exp(x_{n+1}^3 - y_{n+1}^*)]$,

where $y_{n+1}^\star = y_n + 0.2\,\exp(x_n^3 - y_n)$. Now $y_1^\star \approx 1.073575888$; $y(0.2) \approx y_1 \approx 1.071240883$.

$y_2^\star = 1.140307844$, $y(0.4) \approx y_2 \approx 1.139859532$, $y_3^\star \approx 1.208060437$, $y(0.6) \approx y_3 \approx 1.211041171$.

(f) By separating variables, one gets $\frac{dy}{y^2} = -\sin x\,dx$ so $-\frac{1}{y} = \cos x + C$. Now, $y = -[\cos x + C]^{-1}$.

The initial condition yields $C = -3$ and $\boxed{y = \frac{1}{3 - \cos x}}$. **(g)** -23 and $-(-3)\cdot(-23) = -69$.

(h) By the Extreme-Value Theorem, $\exists\ A, B \in [a,b]\ :\ f(A) = \min\limits_{[a,b]} f$ and $f(B) = \max\limits_{[a,b]} f$. Furthermore, applying $\int_a^b dx$ to $f(A) \le f \le f(B)$ yields $f(A) \le \bar{f} \le f(B)$, where $\bar{f} = (b-a)^{-1}\int_a^b f(x)\,dx$.

Finally, by the Intermediate-Value Theorem, $\exists\ c$ between A and B such that $f(c) = \bar{f}$.

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2.(a) Cylindrical shell area: $2\pi x[(e^2-1)-(e^x-1)]$.

$V = 2\pi\int_0^2 x[e^2-e^x]\,dx = 2\pi[\frac{1}{2}x^2e^2 - xe^x + e^x]\Big|_0^2 = 2\pi(e^2-1) \approx 40.14362342.$

(b) (c) $y'(x) = -\dfrac{4x}{4-x^2}.\qquad \sqrt{1+y'^2} = \dfrac{4+x^2}{4-x^2}.$

Arc-length: $\quad s = \displaystyle\int_0^1 \left[-1+\frac{8}{4-x^2}\right]dx = -x+2\ln(2+x)-2\ln(2-x)\Big|_0^1 = -1+2\ln 3 \approx 1.197224578.$

(c) $\rho = \frac{1}{x^2+4x+4} = \frac{1}{(x+2)^2}$; $x\rho = \frac{x}{x^2+4x+4} = \frac{1}{x+2} - \frac{2}{(x+2)^2}$. Center of mass: $\boxed{\bar{x} = M/m \approx 1.054302437}$.

Mass: $m = \int_0^3 \rho\,dx = -\frac{1}{x+2}\big|_0^3 = 0.3.$

Moment: $M = \int_0^3 x\rho\,dx = \ln(x+2) + \frac{2}{x+2}\big|_0^3 = [\ln 5 - \ln 2 - \frac{3}{5}] \approx 0.316290731.$

3.(a) **(i)** Roots: -1, -3 so $y = C_1e^{-x} + C_2\,e^{-3x}$.

(ii) Roots: $-2\pm 3i$ so $y = [C_1\cos(3x) + C_2\sin(3x)]\,e^{-2x}$.

(b) Look for a particular solution $y_p = A + B\,x\,e^{-x}$, which yields $y_p = 3 - 2\,x\,e^{-x}$.

General solution: $y = 3 - 2\,x\,e^{-x} + C_1e^{-x} + C_2\,e^{-3x}$. **(c)** $y = 3 - 2\,x\,e^{-x} - 4e^{-x} - e^{-3x}$.

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4.(a) Answer: $f(h,k) \approx \frac{1}{2} - \frac{1}{4}h - \frac{1}{8}h^2 + \frac{1}{2}kh - \frac{1}{4}k^2$.

$f_x = \frac{-(x+2)\sin(x-y)-\cos(x-y)}{(x+2)^2} = -\frac{\sin(x-y)}{x+2} - \frac{\cos(x-y)}{(x+2)^2}$,

$f_{xx} = -\frac{(x+2)\cos(x-y)-\sin(x-y)}{(x+2)^2} + \frac{(x+2)^2\sin(x-y)+2(x+2)\cos(x-y)}{(x+2)^4}$,

$f_y = \frac{\sin(x-y)}{x+2}$, $\quad f_{xy} = \frac{(x+2)\cos(x-y)-\sin(x-y)}{(x+2)^2}$, $\quad f_{yy} = -\frac{\cos(x-y)}{x+2}$;

$f(0,0) = \frac{1}{2}$, $f_x(0,0) = -\frac{1}{4}$, $f_y(0,0) = 0$, $f_{xx}(0,0) = -\frac{1}{4}$, $f_{xy}(0,0) = \frac{1}{2}$, $f_{yy}(0,0) = -\frac{1}{2}$.

(b) $n = 5$, $(\ln x, \ln y) \approx$ (0., 1.098612289), (0.69314718, 1.6094379), (1.098612289, 2.302585093), (1.386294361, 2.564949357), (1.609437912, 3.091042453). $\sum_{k=1}^{5} \ln x_k \approx 4.787491743$, $\sum_{k=1}^{5} (\ln x_k)^2 \approx 6.199504424$, $\sum_{k=1}^{5} \ln y_k \approx 10.66662710$, $\sum_{k=1}^{5} \ln x_k \cdot \ln y_k \approx 12.17584137$.

$$\beta \approx \frac{n \cdot (12.17584137) - (4.787491743) \cdot (10.66662710)}{n \cdot (6.199504424) - (4.787491743)^2} \approx \boxed{1.214841793},$$

$$\ln k \approx \frac{(10.66662710) - \beta \cdot (4.787491743)}{n} \approx 0.9701164094, \qquad \text{so } k = e^{\ln k} \approx \boxed{2.638251559}.$$

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5.(a) (i) From the last row of the RRE form of this system $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 7 & -7 \\ 0 & 0 & 0 & 2 \end{array}\right]$ one concludes that there are $\boxed{\text{NO solutions}}$

(ii) This system can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $x = [-t, 3+t, t]^T$.

(b) From $\left[\begin{array}{rrrr|rrrr} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & -1 & 5 & 2 & 0 & 1 & 0 & 0 \\ -2 & -2 & 9 & 6 & 0 & 0 & 1 & 0 \\ 8 & 3 & -15 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$ get $\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -143 & 45 & -5 & 12 \\ 0 & 0 & 1 & 0 & -25 & 8 & -1 & 2 \\ 0 & 0 & 0 & 1 & -10 & 3 & 0 & 1 \end{array}\right]$,

and then $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ -143 & 45 & -5 & 12 \\ -25 & 8 & -1 & 2 \\ -10 & 3 & 0 & 1 \end{bmatrix}$.

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