MA4002 Final Exam Answers, Spring 2015

1.(a) Velocity: $v(t) = 3 + \int_0^t \frac{1}{(s+2)^2} ds = 3 + \frac{1}{2} - \frac{1}{t+2}$.
Distance $s(T) = \int_0^T v(t)dt = 3.5\,T + \ln 2 - \ln(T+2)$m and $s(4) = 14 - \ln 3 \approx \boxed{12.901\,\mathrm{m}}$.
(b) Intercepts: $x = 0, 2$. Using cylindrical shells:
$V = \int_0^2 2\pi x\,[(3x - x^2) - x]\,dx = \int_0^2 2\pi(\frac{2}{3}x^3 - \frac{1}{4}x^4)\,dx = \frac{8\pi}{3} \approx 8.37758$.
(c) Integrating by parts using $u = (\ln x)^n$ and $dv = x^2\,dx$ yields the reduction formula
$I_n = \int_1^e x^2\,(\ln x)^n\,dx = \frac{1}{3}x^3\,(\ln x)^n|_1^e - \frac{n}{3} \cdot I_{n-1} = \boxed{\frac{1}{3}e^3 - \frac{n}{3} \cdot I_{n-1}}$. Next, $I_0 = \frac{1}{3}e^3 - \frac{1}{3} \approx 6.3618456$ implies $I_1 = \frac{1}{3}e^3 - \frac{1}{3} \cdot I_0 = \frac{1}{9} + \frac{2}{9}e^3 \approx 4.57456$, and $I_2 = \frac{1}{3}e^3 - \frac{2}{3} \cdot I_1 = -\frac{2}{27} + \frac{5}{27}e^3 \approx 3.6454698$.
(d) $f_x = 3x^2\,e^y$, $f_y = (x^3 - y - 1)\,e^y$, $f_{xx} = 6x\,e^y$, $f_{yy} = (x^3 - y - 2)\,e^y$, $f_{xy} = 3x^2\,e^y$.
(e) $x_n = 0.1n$. Start with $y_0 = 3$. $y_{n+1} = y_n + \frac{1}{2}0.1\,[\ln(x_n^2 + y_n) + \ln(x_{n+1}^2 + y_{n+1}^\star)]$, where $y_{n+1}^\star = y_n + 0.1\,\ln(x_n^2 + y_n)$. Now $y_1^\star \approx 3.109861229$; $y(0.1) \approx y_1 \approx 3.111820041$. $y_2^\star = 3.225661659$, $y(0.2) \approx y_2 \approx 3.227913970$.
(f) Rewrite as $y' - \frac{2}{x}y = -2x^3$ so the integrating factor: $v = \exp\{-\int \frac{2}{x}\,dx\} = x^{-2}$. So $(x^{-2} \cdot y)' = -2x$ and therefore $x^{-2} \cdot y = -x^2 + C$ so $\boxed{y = -x^4 + Cx^2}$. By $y(1) = 5$ we get $C = 6$ and $\boxed{y = 6x^2 - x^4}$.
(g) 39 and $-(-2) \cdot 39 = 78$.
(h) For $x > 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$, while for $x < 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-x)' = \frac{1}{x}$. Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$. The desired result follows.

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2.(a) The cross-sectional area is $\pi\,x^{2/3}$, so when the depth of liquid is c, the volume is $V(c) = \int_0^c \pi\,x^{2/3}\,dx = \frac{3}{5}\pi c^{5/3}$. So (i) the total volume of the glass is $V(8) = \frac{3}{5}\pi 8^{5/3} = \frac{3}{5}\pi\,2^5 = \frac{96}{5}\pi \approx 60.3185$; (ii) the glass is 70% full if $V(c) = 0.7 \cdot V(8)$ or $c^{5/3} = 0.7 \cdot 8^{5/3}$ or $c = 8 \cdot 0.7^{3/5} \approx 6.458755$.
(b) $x'(t) = -4\sin(4t)$, $y'(t) = -4\cos(4t)$, $z'(t) = -3$; $\sqrt{x'^2 + y'^2 + z'^2} = \sqrt{4^2 + 3^2} = 5$.
Arc-length: $= \int_0^7 5\,dt = 5t|_0^7 = 35$.
(c) $\rho = \frac{1}{x^2-4} = \frac{1}{4}(\frac{1}{x-2} - \frac{1}{x+2})$; $x\rho = \frac{x}{x^2-4} = \frac{1}{2}\frac{(x^2-4)'}{x^2-4}$. Center of mass: $\boxed{\bar{x} = M/m \approx 3.76593}$.
Mass: $m = \int_3^5 \rho\,dx = \frac{1}{4}[\ln(x-2) - \ln(x+2)]|_3^5 = \frac{1}{4}(\ln 5 + \ln 3 - \ln 7) \approx 0.190535$. Moment: $M = \int_3^5 x\rho\,dx = \frac{1}{2}\ln(x^2-4)|_3^5 = \frac{1}{2}[-\ln 5 + \ln 3 + \ln 7] \approx 0.717542$.

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3.(a) **(i)** Roots: $0, -7$ so $y = C_1 + C_2\,e^{-7x}$. **(ii)** Roots: $1 \pm 2i$ so $y = [C_1\,\cos(2x) + C_2\,\sin(2x)]\,e^x$.
(b) Look for a particular solution $y_p = A\,x + B\,e^{-2x}$, which yields $y_p = x + \frac{1}{2}\,e^{-2x}$.
General solution: $y = x + \frac{1}{2}\,e^{-2x} + C_1 + C_2\,e^{-7x}$. **(c)** $y = x + \frac{1}{2}\,e^{-2x} + \frac{1}{2} - 3\,e^{-7x}$.

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4.(a) Answer: $f(h,k) \approx 1 + 2h + 2k + \frac{3}{2}\,h^2 + 4\,kh + 2\,k^2$.
$f_x = (x+2)\,e^{x+2y}$, $f_{xx} = (x+3)\,e^{x+2y}$, $f_y = 2(x+1)\,e^{x+2y}$, $f_{xy} = 2(x+2)\,e^{x+2y}$, $f_{yy} = 4(x+1)\,e^{x+2y}$;
$f_x(0,0) = 2 \cdot 1 = 2$, $f_{xx}(0,0) = 3 \cdot 1 = 3$, $f_y(0,0) = 2 \cdot 1 = 2$, $f_{xy}(0,0) = 4 \cdot 1 = 4$, $f_{yy}(0,0) = 4 \cdot 1 = 4$.

(b) $n = 4$, $(\ln x, \ln y) \approx (0.693147,\ 2.397895), (1.38629,\ 2.89037), (1.791759,\ 2.7725887), (2.0794415,\ 3.091042)$.
$\sum_{k=1}^{4} \ln x_k \approx 5.95064$, $\sum_{k=1}^{4} (\ln x_k)^2 \approx 9.936744$, $\sum_{k=1}^{4} \ln y_k \approx 11.151898$, $\sum_{k=1}^{4} \ln x_k \cdot \ln y_k \approx 17.06445$.

$$\beta \approx \frac{n \cdot (17.06445) - (5.95064) \cdot (11.151898)}{n \cdot (9.936744) - (5.95064)^2} \approx \boxed{0.4373836},$$

$$\ln k \approx \frac{(11.151898) - \beta \cdot (5.95064)}{n} \approx 2.137296, \qquad \text{so } k = e^{\ln k} \approx \boxed{8.4764876}.$$

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5.(a) (i) This system can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $\boxed{x = [9, -4, -2]^T}$. **(ii)** From the last row of the RRE form of this system $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & -2 & 0 & 8 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 17 \end{array}\right]$ one concludes that there are $\boxed{\text{NO solutions}}$

(b) From $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -2 & 3 & -1 & 0 & 1 & 0 & 0 \\ -5 & 8 & -9 & 7 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$ get $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{9}{2} & \frac{15}{2} & \frac{5}{2} & 2 \\ 0 & 0 & 1 & 0 & -3 & \frac{13}{3} & \frac{4}{3} & 1 \\ 0 & 0 & 0 & 1 & 2 & -3 & -1 & -1 \end{array}\right]$,
and then $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{9}{2} & \frac{15}{2} & \frac{5}{2} & 2 \\ -3 & \frac{13}{3} & \frac{4}{3} & 1 \\ 2 & -3 & -1 & -1 \end{bmatrix}$.

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