MA4002 Final Exam Answers, Spring 2012

1.(a) Velocity: $v(t) = 0 + \int_0^t (20 + 30\sqrt{s})ds = 20t + 20t^{3/2}$.
Distance $s(T) = \int_0^T v(t)dt = 10T^2 + 8T^{5/2}$m and $s(4) = 160 + 8 \cdot 2^5 = \boxed{416\,\text{m}}$.
(b) (i) The cross-sectional area: $\pi(\frac{1}{x+1})^2$. $V = \pi \int_0^2 1(\frac{1}{x+1})^2\,dx = \pi(-\frac{1}{x+1})|_0^2 = \frac{2}{3}\pi \approx 2.094395$.
(ii) Using cylindrical shells: $V = \int_0^2 2\pi x(\frac{1}{x+1})\,dx = \int_0^2 2\pi(1 - \frac{1}{x+1})\,dx = 2\pi(x - \ln|x+1|)|_0^2 = 2\pi(2 - \ln 3) \approx 5.663586$.
(c) Integrating by parts using $u = (x+1)^n$ and $dv = e^{-x/3}\,dx$ yields the reduction formula $I_n = \int_0^3 (x+1)^n\, e^{-x/3}\,dx = -3(x+1)^n\, e^{-x/3}|_0^3 + 3n \cdot I_{n-1} = \boxed{3 - 3\cdot 4^n \cdot e^{-1} + 3n \cdot I_{n-1}}$. Next, $I_0 = 3 - 3\,e^{-1} \approx 1.89636$ implies $I_1 = 3 - 12\,e^{-1} + 3\,I_0 = 12 - 21\,e^{-1} \approx 4.2745317$ and $I_2 = 3 - 48\,e^{-1} + 6\,I_1 = 75 - 174\,e^{-1} \approx 10.988977$. **(d)** $f_x = \frac{1}{x+y^2}$, $f_y = \frac{2y}{x+y^2}$, $f_{xx} = \frac{-1}{(x+y^2)^2}$, $f_{yy} = 2\frac{x-y^2}{(x+y^2)^2}$, $f_{xy} = \frac{-2y}{(x+y^2)^2}$.
(e) $x_n = 0.1n$. Start with $y_0 = 3$. $y_{n+1} = y_n + \frac{1}{2}0.1\,[\sqrt{x_n^3 + y_n} + \sqrt{x_{n+1}^3 + y_{n+1}^\star}]$, where $y_{n+1}^\star = y_n + 0.1\sqrt{x_n^3 + y_n}$. Now $y_1^\star \approx 3.173205081$; $y(0.1) \approx y_1 \approx 3.175684035$. $y_2^\star = 3.353916581$, $y(0.2) \approx y_2 \approx 3.356477958$.
(f) Rewrite as $y' - \frac{2}{x}y = -2x^3$ so the integrating factor: $v = \exp\{\int(-\frac{2}{x})\,dx\} = x^{-2}$. So $(x^{-2}\cdot y)' = -2x$ and therefore $x^{-2}\cdot y = -x^2 + C$ so $y = -x^4 + C\,x^2$. By $y(1) = 5$ we get $C = 6$ and $\boxed{y = -x^4 + 6\,x^2}$.
(g) 3 and $-(-9)\cdot 3 = 27$.
(h) An integration by parts using $u = f(x)$ and $dv = dx$ with $v = x - x_1$ yields:
$\int_{x_0}^{x_1} f(x)\,dx = f(x)\cdot(x - x_1)|_{x_0}^{x_1} - \int_{x_0}^{x_1}(x - x_1)f'(x)\,dx = 0 - f(x_0)\cdot(-h) - \int_{x_0}^{x_1}(x - x_1)f'(x)\,dx$. The desired relation follows.

2.(a) The glass height is $e - 1$ and using cylindrical shell area $2\pi x[(e-1) - (e^x - 1)] = 2\pi x[e - e^x]$, one gets $V = \int_0^1 2\pi x[e^x - e]\,dx = 2\pi(\frac{1}{2}\,e\,x^2 - x\,e^x + e^x)\Big|_0^1 = \boxed{2\pi\,(\frac{1}{2}\,e - 1) \approx 2.2565489}$.
(b) $x'(t) = [\cos(2t) - 2\sin(2t)]\,e^t$, $y'(t) = [\sin(2t) + 2\cos(2t)]\,e^t$; $\sqrt{x'^2 + y'^2} = \sqrt{5}\,e^t$.
Arc-length: $= \int_0^\pi \sqrt{5}\,e^t\,dt = \sqrt{5}\,[e^\pi - 1] \approx 49.50809380$.
(c) $\rho = \frac{1}{(x+2)^2}$; $x\rho = \frac{x}{(x+2)^2} = \frac{1}{x+2} - \frac{2}{(x+1)^2}$. Center of mass: $\bar{x} = M/m = \frac{\ln 5 - \ln 2 - \frac{3}{5}}{0.3} \approx 1.054302437$.
Mass: $m = \int_0^3 \rho\,dx = [-\frac{1}{x+2}]|_0^3 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10} = 0.3$. Moment: $M = \int_0^3 x\rho\,dx = [\ln|x+2| + \frac{2}{x+2}]|_0^3 = \ln 5 - \ln 2 - \frac{3}{5} \approx .3162907314$.

3.(a) (i) Roots: $-3\pm 4\,i$ so $y = [C_1\cos(4x) + C_2\sin(4x)]\,e^{-3x}$. **(ii)** Roots: 0, -3 so $y = C_1 + C_2e^{-3x}$.
(b) Look for a particular solution $y_p = a\,x + b\,\sin x + c\,\cos x$, which yields $y_p = -2x - \sin x - 3\cos x$.
General solution: $y = -2x - \sin x - 3\cos x + C_1 + C_2e^{-3x}$. **(c)** $y = -2x - \sin x - 3\cos x + 7 - e^{-3x}$.

4.(a) Answer: $f(1+h, 3+k) \approx 8 + 6h + 3k + \frac{15}{4}h^2 + \frac{3}{4}hk + \frac{3}{16}k^2$.
$f_x = 3x\sqrt{x^2+y}$, $f_{xx} = \frac{6x^2+3y}{\sqrt{x^2+y}}$, $f_y = \frac{3}{2}\sqrt{x^2+y}$, $f_{yy} = \frac{3}{4\sqrt{x^2+y}}$, $f_{xy} = \frac{3x}{2\sqrt{x^2+y}}$;
$f_x(1,3) = 3\cdot 1\cdot 2 = 6$, $f_{xx}(1,3) = \frac{6\cdot 1^2 + 3\cdot 3}{2} = \frac{15}{2}$, $f_y(1,3) = \frac{3}{2}\cdot 2 = 3$, $f_{yy}(1,3) = \frac{3}{4\cdot 2} = \frac{3}{8}$, $f_{xy}(1,3) = \frac{3\cdot 1}{2\cdot 2} = \frac{3}{4}$.

(b) $n = 5$, $(\ln x, \ln y) \approx (-0.6931, 0), (0, 0.9555), (0.6931, 2.3979), (1.0986, 3.4012), (1.3863, 3.9120)$.
$\sum_{k=1}^{5}\ln x_k \approx 2.4849$, $\sum_{k=1}^{5}(\ln x_k)^2 \approx 4.089667$, $\sum_{k=1}^{5}\ln y_k \approx 10.66662$, $\sum_{k=1}^{5}\ln x_k\cdot\ln y_k \approx 10.821907$.
$\alpha \approx \frac{n\cdot(10.821907) - (2.4849)\cdot(10.66662)}{n\cdot(4.089667) - (2.4849)^2} \approx \boxed{1.9339}$, $\ln k \approx \frac{(10.66662) - \alpha\cdot(2.4849)}{n} \approx 1.1722$,
so $k = e^{\ln k} \approx \boxed{3.2291}$.

5.(a) (i) $x = [-5, 1, 8]^T$. **(ii)** From $\left[\begin{array}{rrr|r} 2 & 0 & 1 & -2 \\ 0 & 2 & -1 & -6 \end{array}\right]$ obtain $x = [-1 - \frac{1}{2}\,t_1,\ -3 + \frac{1}{2}\ t_1, t_1]^T$.

(b) From $\left[\begin{array}{rrrr|rrrr} 2 & -3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & -5 & 1 & -1 & 0 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 7 & 0 & 0 & 0 & 1 \end{array}\right]$ get $\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 35 & -21 & -\frac{15}{2} & -3 \\ 0 & 1 & 0 & 0 & 23 & -14 & -5 & -2 \\ 0 & 0 & 1 & 0 & -35 & 21 & 7 & 3 \\ 0 & 0 & 0 & 1 & -10 & 6 & 2 & 1 \end{array}\right]$,

and then $A^{-1} = \begin{bmatrix} 35 & -21 & -\frac{15}{2} & -3 \\ 23 & -14 & -5 & -2 \\ -35 & 21 & 7 & 3 \\ -10 & 6 & 2 & 1 \end{bmatrix}$.