MA4002 Final Exam Answers, Spring 2010

1.(a) Velocity: $v(t) = 5 + \int_0^t \cos(0.25s)\,ds = 5 + 4\sin(0.25t)$. Distance $s(T) = \int_0^T v(t)dt = 5T + 16 - 16\cos(0.25T)$ m and $s(10) = \boxed{66 - 16\cos(2.5) \approx 78.81829785\,\text{m}}$.

(b) The cross-sectional area: $\frac{\pi}{(\sqrt{\sin(\pi x)})^2}$. $\quad V = \pi \int_0^1 \sin(\pi x)\,dx = \pi\Big(-\frac{1}{\pi}\cos(\pi x)\Big)\Big|_0^1 = 2.$

(c) Integrating by parts yields the reduction formula $I_n = -2x^n\,e^{-x/2}\big|_0^2 + 2nI_{n-1} = -2^{n+1}\,e^{-1} + 2nI_{n-1}$. Next, $I_0 = 2 - 2e^{-1} \approx 1.26424$ implies $I_1 = 4 - 8e^{-1} \approx 1.05696$, $I_2 = 16 - 40e^{-1} \approx 1.28482$ and $I_3 = 96 - 256e^{-1} \approx 1.82286$.

(d) $f_x = [1-y]e^{x-xy}$, $f_y = -x\,e^{x-xy}$, $f_{xx} = [1-y]^2e^{x-xy}$, $f_{yy} = x^2e^{x-xy}$, $f_{xy} = [-1-x+xy]e^{x-xy}$.

(e) $x_n = 0.1n$. Start with $y_0 = 1$. $\quad y_{n+1} = y_n + \frac{1}{2}0.1[e^{x_n - y_n^2} + e^{x_{n+1}-(y^\star_{n+1})^2}]$, where $y^\star_{n+1} = y_n + 0.1e^{x_n - y_n^2}$. Now $y_1^\star = 1 + .1\cdot e^{-1} \approx 1.036787944$; $\quad y(0.1) \approx y_1 \approx 1.037254924$. $\quad y_2^\star = 1.074940311$, $y(0.2) \approx y_2 \approx 1.075328671$. $\quad y_3^\star \approx 1.113758672$, $\quad y(0.3) \approx y_3 \approx 1.114066113$.

(f) By separating variables, one gets $\frac{dy}{y} = -\frac{x}{x+1}\,dx$ so $\ln|y| = -\int \frac{x}{x+1}\,dx = \int(\frac{1}{x+1} - 1)dx = \ln|x+1| - x + C$. Now, $|y| = e^C|x+1|e^{-x}$ or $y = C'(x+1)\,e^{-x}$, where C' is an arbitrary constant. The initial condition yields $\boxed{y = 5(x+1)\,e^{-x}}$. $\quad$ **(g)** 13.

(h) For $x > 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$, while for $x < 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-x)' = \frac{1}{x}$. Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$. The desired result follows.

2.(a) Cylindrical shell area: $2\pi x[8 - x^3]$. $\quad V = \int_0^2 2\pi x[8-x^3]dx = 2\pi(4x^2 - \frac{x^5}{5})\Big|_0^2 = \frac{96\pi}{5} \approx 60.32.$

(b) $y'(x) = \sinh x$; $\sqrt{1+y'^2} = \cosh x$. Arc-length $= \int_0^3 \cosh x\,dx = \sinh x\Big|_0^3 = \sinh 3 \approx 10.01787.$

(c) $\rho = \dfrac{1}{4-x^2} = \dfrac{1}{4}\Big(\dfrac{1}{x+2} - \dfrac{1}{x-2}\Big)$; $\quad x\rho = \dfrac{x}{4-x^2} = \dfrac{1}{2}\Big(-\dfrac{1}{x-2} - \dfrac{1}{x+2}\Big)$. Center of mass: $\bar{x} = M/m = 2\dfrac{2\ln 2 - \ln 3}{\ln 3} \approx .523719$. Mass: $m = \displaystyle\int_0^1 \rho\,dx = \frac{1}{4}[\ln(x+2) - \ln|x-2|]\Big|_0^1 = \frac{1}{4}\ln 3 \approx .274653.$
Moment: $M = \displaystyle\int_0^1 x\rho\,dx = \frac{1}{2}[-\ln|x-2| - \ln(x+2)]\Big|_0^1 = \frac{1}{2}[2\ln 2 - \ln 3] \approx .143841.$

3.(a) **(i)** $y = [C_1\cos(2x) + C_2\sin(2x)]\,e^x$. $\quad$ **(ii)** $y = C_1e^x + C_2e^{5x}$.

(b) Look for a particular solution in the form $y_p = ax^2 + bx + c$, which yields $y_p = 5x^2 + 4x - 0.4$. General solution: $y = 5x^2 + 4x - 0.4 + [C_1\cos(2x) + C_2\sin(2x)]\,e^x$.

(c) $y = 5x^2 + 4x - 0.4 + [2\cos(2x) - \sin(2x)]\,e^x$.

4.(a) Answer: $f(h, 1+k) \approx \frac{1}{3} - \frac{1}{9}h - \frac{1}{3}k - \frac{7}{54}h^2 + \frac{2}{9}hk + \frac{1}{3}k^2$.

$f_x = -\frac{\sin x}{x+3y} - \frac{\cos x}{(x+3y)^2}$, $\quad f_{xx} = -\frac{\cos x}{x+3y} + \frac{2\sin x}{(x+3y)^2} + \frac{2\cos x}{(x+3y)^3}$,

$f_y = -\frac{3\cos x}{(x+3y)^2}$, $\quad f_{yy} = \frac{18\cos x}{(x+3y)^3}$, $\quad f_{xy} = \frac{3(x+3y)\sin x + 6\cos x}{(x+3y)^3}$.

(b) $n = 5$, $\displaystyle\sum_{k=1}^5 x_k = 5$, $\displaystyle\sum_{k=1}^5 x_k^2 = 15$, $\displaystyle\sum_{k=1}^5 y_k = 2 + 3A + B$, $\displaystyle\sum_{k=1}^5 x_ky_k = 12 + A + 2B$.

$$a = \frac{n\cdot(12 + A + 2B) - 5\cdot(2 + 3A + B)}{n\cdot 15 - 5^2} = \frac{10 - 2A + B}{10}, \qquad b = \frac{(2 + 3A + B) - a\cdot 5}{n} = \frac{-3 + 3A + B}{5}.$$

Combining this with $a = 1$ and $b = 0$ yields the Answer: $\boxed{A = \frac{3}{5},\ B = \frac{6}{5}}$.

5.(a) **(i)** $x = [11, 0, 3]^T$. **(ii)** From $\left[\begin{array}{ccr|r} 1 & 0 & 8 & 35 \\ 0 & 1 & -\frac{3}{2} & -\frac{9}{2} \end{array}\right]$ obtain $x = [35 - 8t,\ -\frac{9}{2} + \frac{3}{2}t,\ t]^T$.

(iii) From $\left[\begin{array}{cccr|r} 1 & 0 & -1 & -19 & -3 \\ 0 & 1 & -2 & -4 & -1 \end{array}\right]$ obtain $x = [-3 + t_1 + 19t_2,\ -1 + 2t_1 + 4t_2,\ t_1,\ t_2]^T$.

(b) $A^{-1} = \begin{bmatrix} -17 & \frac{11}{2} & -1 \\ \frac{13}{2} & -2 & \frac{1}{2} \\ \frac{19}{2} & -3 & \frac{1}{2} \end{bmatrix}$.