MA4002 Final Exam Answers, Spring 2004

1.(a) $v = 15 - 5e^{-0.2t}$. $\quad s = (15t + 25e^{-0.2t})\Big|_0^{10} = 125 + 25e^{-2} \approx 128.38$.

(b) $3 \cdot 2^{1/3}$.

(c) The cross-sectional area: $\pi(\sqrt{\sin x})^2 = \pi \sin x$. $\quad V = \pi \int_0^\pi \sin x \, dx = 2\pi$.

(d) Reduction formula: $I_n = -e^{-1} + nI_{n-1}$.

$I_0 = \int_0^1 e^{-x}\, dx = 1 - e^{-1}$. $\quad I_1 = -e^{-1} + I_0 = 1 - 2e^{-1}$, $\quad I_2 = -e^{-1} + 2I_1 = 2 - 5e^{-1}$.

(e) $f_x = ye^{xy}$, $\quad f_y = xe^{xy}$, $\quad f_{xx} = y^2 e^{xy}$, $\quad f_{yy} = x^2 e^{xy}$, $\quad f_{xy} = (1 + xy)\, e^{xy}$.

(f) $x_n = 0.4n$. Start with $y_0 = 1$. $\quad y_{n+1} = y_n + 0.2(x_n y_n + x_{n+1} y^\star_{n+1})$, where $y^\star_{n+1} = y_n + 0.4 x_n y_n$.

$y^\star_1 = 1$, $\quad y(0.4) \approx y_1 = 1 + 0.2(0 + 0.4 \times 1) = 1.08$.

(g) Separable variables: $\dfrac{dy}{y} = \dfrac{2x\, dx}{1 + x^2}$ implies that $y = C(1 + x^2)$. Solution: $y = -(1 + x^2)$.

(h) -11.

2.(a) Area: $\quad \int_0^1 \dfrac{2x}{1 + x^2}\, dx = \ln(1 + x^2)\Big|_0^1 = \ln 2 \approx 0.69$.

(b) Cylindrical shell area: $2\pi x \cos x$. $\quad V = 2\pi \int_0^{\pi/2} x \cos x \, dx = 2\pi(\cos x + x \sin x)\Big|_0^{\pi/2} = 2\pi(\pi/2 - 1)$.

(c) $y'(x) = \dfrac{1}{2x} - \dfrac{x}{2}$. $\sqrt{1 + y'^2} = \dfrac{1 + x^2}{2x}$. Arc-length: $\quad s = \int_1^e \dfrac{1 + x^2}{2x}\, dx = \Big(\dfrac{\ln x}{2} + \dfrac{x^2}{4}\Big)\Big|_1^e = \dfrac{e^2 + 1}{4}$.

(d) $\rho = \dfrac{1}{x + 1} - \dfrac{1}{x + 2}$, while $x\rho = \dfrac{2}{x + 2} - \dfrac{1}{x + 1}$. Mass: $\quad m = \int_0^2 \rho\, dx = \ln 3 - \ln 2 \approx 0.405$.

Moment: $\quad M = \int_0^2 x\rho\, dx = 2\ln 2 - \ln 3 \approx 0.288$. Center of mass: $\quad \bar{x} = M/m = \dfrac{2\ln 2 - \ln 3}{\ln 3 - \ln 2} \approx 0.71$.

3.(a) **(i)** $y = C_1 e^{-x} + C_2 e^{-2x}$. $\quad$ **(ii)** $y = C_1 \sin(2x) + C_2 \cos(2x)$.

(b) Particular solution: $y_p = 2x^2 - 6x + 7$. $\quad$ General solution: $y = 2x^2 - 6x + 7 + C_1 e^{-x} + C_2 e^{-2x}$.

(c) $y = 2x^2 - 6x + 7 - 2e^{-x} - 3e^{-2x}$.

4.(a) Answer: $f(x, y) \approx 1 + y - x + x^2 + y^2/2$.

$f_x = \Big[\dfrac{y}{x + 1} - \dfrac{1}{(x + 1)^2}\Big] e^{y+xy}$, $\qquad f_{xx} = \Big[\dfrac{y^2}{x + 1} - \dfrac{2y}{(x + 1)^2} + \dfrac{2}{(x + 1)^3}\Big] e^{y+xy}$,

$f_y = e^{y+xy}$, $\qquad f_{xy} = y\, e^{y+xy}$, $\qquad f_{yy} = (x + 1)\, e^{y+xy}$.

(b) $n = 5$, $\quad \sum_{k=1}^5 x_k = 5$, $\quad \sum_{k=1}^5 x_k^2 = 15$, $\quad \sum_{k=1}^5 y_k = 10$, $\quad \sum_{k=1}^5 x_k y_k = 22$.

$a = \dfrac{n \cdot 22 - 5 \cdot 10}{n \cdot 15 - 5^2} = 1.2$, $\quad b = \dfrac{10 - a \cdot 5}{n} = 0.8$. $\quad$ Answer: $y = 1.2x + 0.8$.

5.(a) $AA^T = \begin{bmatrix} 67 & -18 \\ -18 & 14 \end{bmatrix}$.

(b) **(i)** $x = [2,\, -3,\, 6]^T$. $\quad$ **(ii)** From $\left[\begin{array}{ccc|c} 1 & 0 & -4 & -22 \\ 0 & 1 & 2 & 9 \end{array}\right]$ obtain $\quad x = [-22 + 4t,\, 9 - 2t,\, t]^T$.

(c) $A^{-1} = \dfrac{1}{2}\begin{bmatrix} 19 & 25 & -11 \\ -5 & -7 & 3 \\ 3 & 3 & -1 \end{bmatrix}$.