MA4002 Final Exam Solutions 1996

1.(i) $v = 2t + 10t^{\frac{3}{2}}$; $\quad s = t^2 + 4t^{\frac{5}{2}}$.

(ii) Integrate by parts with $u = \ln x$ and $dv = \sqrt{x}\,dx$. Answer: $\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$.

(iii) $2x\tan(x^2)$. **(iv)** $\dfrac{1}{4-1}\displaystyle\int_1^4 \frac{1}{x}\,dx = \frac{1}{3}\ln 4$. **(v)** Try it yourself!

(vi) $\displaystyle\int_0^2 \cos x\,dx = \sin 2$. **(vii)** $e_Q = 3e_x + \dfrac{y}{y+1}e_y$. **(viii)** Variables separable. $y = \tan(\sin x)$.

(ix) $y_{n+1} = y_n + 0.1(0.01n^2 + y_n^2)$, where $y_0 = 2$. **(x)** -4.

2.(i) Multiply above and below by $\sec x + \tan x$, and then substitute $u = \sec x + \tan x$.
Answer: $\ln|\sec x + \tan x| + C$. **(ii)** Substitute $u = x + 1$ to obtain $\displaystyle\int_1^\infty \frac{1}{u^2+3}\,du = \frac{\pi}{3\sqrt{3}}$.

(iii) Substitute $u = \tan(\frac{t}{2})$ to obtain

$$\int_0^{\sqrt{2}-1} \frac{1}{(2u-1)(u-2)}\,du = \frac{1}{3}\int_0^{\sqrt{2}-1}\left(\frac{1}{u-2} - \frac{1}{u-\frac{1}{2}}\right)du = \frac{1}{3}\left(\ln(3-\sqrt{2}) - \ln(3-2\sqrt{2}) - \ln 2\right).$$

3. (i) $\displaystyle\int_0^\infty (\cosh x - \sinh x)\,dx = \int_0^\infty e^{-x}\,dx = 1$. (ii) By disks $V = \displaystyle\int_0^2 \pi\frac{9}{(y+1)^2}\,dy = 6\pi$.

(iii) $M = \displaystyle\int_0^4 \rho_0(1+x)\,dx = 12\rho_0$. $I = \displaystyle\int_0^4 \rho_0(1+x)(x-2)^2\,dx = 16\rho_0$.

(iv) $s = \displaystyle\int_0^2 \sqrt{36t^2 + (1-9t^2)^2}\,dt = \int_0^2 (1+9t^2)\,dt = 26$.

4.(a) $f(\frac{\pi}{2}+h, k) = \frac{\pi}{2} + 1 + h + \pi k - \frac{1}{2}h^2 + 2hk + \pi k^2 + \cdots$.

(b) $\sum x_i = 10$, $\sum y_i = 13 + b$, $\sum x_i^2 = 30$, $\sum x_i y_i = 34 + 3b$. So $m = \dfrac{11}{10} = \dfrac{1}{10}(8+b)$. Hence $b = 3$.

5.(a) Integrate by parts with $u = \cos^{2n-1} x$ and $dv = \cos x\,dx$, and use $\sin^2 x = (1 - \cos^2 x)$ in remaining integral to obtain $I_n = (2n-1)I_{n-1} - (2n-1)I_n$.
Hence $I_n = \dfrac{2n-1}{2n}I_{n-1} = \cdots = \dfrac{(2n-1)(2n-3)\cdots 5.3.1}{2^n n!}I_0 = \dfrac{(2n)!\pi}{2^{2n+1}n!n!}$.

(b) $h = \frac{1}{4}$. $S_4 \approx 1.18415$. $E_S < \dfrac{5h^4}{9} \approx 2.17 \times 10^{-3}$. $E_S < \dfrac{5}{144n^4} < 5 \times 10^{-21} \Rightarrow 2n > 102669$.

6.(a) Characteristic equation $\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = -2 \pm i$. So $q_h(t) = Ae^{-2t}\cos t + Be^{-2t}\sin t$.

(b) Try $q_p = \alpha\cos t + \beta\sin t$, to find $\alpha = 1$, $\beta = 1$. Hence $q(t) = Ae^{-2t}\cos t + Be^{-2t}\sin t + \cos t + \sin t$.

(c) $q(t) = \cos t + \sin t - e^{-2t}\sin t$.

7.(a) (i) $x + y = 1$, $x + y = 2$, $x + y = 3$. (ii) $x = 1$, $y = 2$, $x + y = 3$.

(iii) $x + y = 1$, $2x + 2y = 2$, $3x + 3y = 3$.

(b)

$$A^{-1} = \begin{bmatrix} -9 & -5 & 6 \\ 12 & 7 & -8 \\ 11 & 6 & -7 \end{bmatrix}.$$