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The two envelope problem: there is no conundrum

S. L. MITCHELL* AND S. B. G. O'BRIEN
Mathematics Applications Consortium for Science and Industry (MACSI),
Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland

**Email: sarah.mitchell@ul.ie*

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We consider the famous two envelope conundrum and show, in a very simple way, that there is no conundrum. Our discussion is supported by numerical simulations. We use the problem to raise the issue of disputes in mathematics, rarely touched upon in mathematics education. We suggest that it is worthwhile to expose students of mathematics to such controversies and that this problem can be introduced in the classroom via an experimental demonstration.

1. Introduction

Students of mathematical modelling (or applied mathematics) are rarely exposed to controversy. Mathematics courses are presented didactically as a series of theorems and results where disputes over correctness never arise. However, in the development of mathematical models at research level, things are often not so black and white. The scientific method involves the development of hypotheses which ultimately must be compared with experiment and observation. Certainly, when one is dealing with mathematical modelling of poorly understood complex processes, the path to the scientific truth is often tortuous, with many false turns and dead ends. Models for global warming would, at present, be examples of disputed models. This is partly due to the complexity and multiscale nature of the processes involved. In addition, even when the fundamental processes are well understood, the physical parameters which quantify the relative sizes of competing effects are often not (yet) known to a sufficient precision. The truth tends to emerge slowly. In theoretical models of the universe, where empirical/observational results are limited, disputes often arise and opposing 'schools' develop. In the absence of empirical support, disciples of the different schools can display an almost religious zeal. 'Pure' (or Platonic) mathematics, in its most extreme form, is an application of logic to the investigation of the properties of a mathematical object. As such, pure mathematics is not regarded as susceptible to the scientific method. But even here, the distinction between theory and empiricism is not always clear and disputes over the correctness of proofs can arise. The proof of the four-colour problem is a case in point: this was achieved via the construction of a computer programme and the testing of each case separately. One might argue that this is not quite an example of theory meeting experiment in mathematics. Nevertheless, it was certainly regarded as controversial at the time.

The occurrence of disputes is perhaps not so surprising in the context of complex mathematical modelling questions. However, disputes can also arise in problems which are apparently cut and dried. In particular, there have been a number of examples in the area of applied probability which have led to disputes. One example of this is the famous Monty Hall problem (Rosenhouse, 2009) (discussed briefly in the Appendix), which now seems to have been resolved. In the present article, we focus on a similar type of problem, the two envelope problem (TEP), which, remarkably, still leads to disputes. In order to understand the problem, one only needs to have a basic understanding of probability distributions and mathematical expectation. It is interesting that, although the problem lends itself to experimental investigation, this is rarely the approach of theoretical investigators from both sides of the argument who, often intolerantly, assert the absolute correctness of their approach. In what follows, we attempt to give our slant on the problem in as simple a manner as possible. In addition, we place emphasis on explaining *why* the disputes arise in the context of this particular problem. In a classroom setting, we suggest that the competing hypotheses could be tested experimentally, probably in the form of the TEP+, introduced below. In the context of this article, we use numerical simulations as a substitute for such experimental demonstrations.

1.1 The problem as a game

The two envelope problem has been discussed since the 1950s and, apparently, it still provokes controversy and disagreement, see, for example, Christensen & Utts (1992, 1993); Broome (1995); Blachman *et al.* (1996); Joyce (1999); Clark & Shackel (2000); Chalmers (2002); Meacham & Weisberg (2003); Devlin (2004); Storkey (2005); McDonnell & Abbott (2009); McDonnell et al. (2011), and references therein.

TEP: single player version It is easy to describe the TEP as a single player game. *There are two envelopes, one holding twice as much money as the other (or containing pieces of papers with positive real numbers written on them where one number is twice the size of the other). The player has no knowledge of what is contained in the envelopes, apart from the fact that one envelope contains twice what the other contains. The player is allowed to open one envelope and, having seen the contents, may either keep that amount or opt to change envelope. Is it better to always stick with the original choice or always change envelope in order to obtain the maximum amount of money (or largest sum of numbers) over many (N) repetitions?*

Our aim in this short article is to demonstrate that the strategies of sticking or changing envelope are equivalent. The apparent conundrum, much discussed in the literature (Devlin, 2004; Storkey, 2005; McDonnell & Abbott, 2009), is based on an incorrect calculation and confuses the TEP with the single envelope problem (SEP), discussed below.

We will give two transparent arguments to illustrate that there is no conundrum and why the standard calculation (1) leading to the conundrum is wrong. We will demonstrate pictorially *why* the problem has been the cause of so much discussion. Finally, we will back up this point of view with numerical simulations. We will initially avoid any detailed discussion of the problem in terms of probabilities, Bayes' theorem (see, for example, Christensen & Utts, 1992; Brams & Kilgour, 1995, 1998; Devlin, 2004) where knowledge of a prior distribution is discussed), which, in our opinion, may confuse our main and simple point. Later, in Section 7, we present a more formal argument to support the intuitive approach of the first part of the article.

In point of fact, much of the confusion is traceable to the related SEP.

SEP: single player version In this case, the player is given a single envelope, allowed to see the contents, and then has the option of sticking with this amount or having the amount either doubled or halved on the basis of a coin toss. What is the best strategy to maximize the total sum over many repetitions? As we will show, despite the apparent similarity, the two problems are quite different. Indeed, the better strategy in the single envelope game is to always make the switch rather than stick with the contents of the envelope.

2. Details

The TEP can be discussed in the context of a random variable, $y \in \mathbb{R}^+$, drawn from some random distribution which is not known to the player of the game. (This is discussed in more detail in Section 8.) In this case, the problem is to develop a strategy to maximize the sum from a sequence of $i = 1 \ldots N$ pairs of envelopes containing the amounts $\{y_i, 2y_i\}$, where y_i is drawn from the aforementioned distribution.

However, there is another way of representing the TEP *as a multiple player game*.

TEP+: TEP, multiple player version *Consider a game where there are* N *players who wish to develop a strategy to maximize the amount obtained by the whole group. Each player is given two envelopes containing the same fixed amounts* $y, 2y$. y *is known to the host of the game but unknown, of course, to the players. Each player does know that one envelope contains twice as much as the other. The players are not allowed to communicate during the game; they are each allowed to open one envelope of their choice and view the contents, and then they are asked if they want to stick or switch.*

The point of describing the problem as a multiple player game is to ensure that each trial (i.e. what each player decides to do with his pair of envelopes) is independent and there need be no mention of any distribution from which y is drawn (y is a constant in this version of the game). Viewing the contents of the first envelope certainly gives no useful information to each player, so the probabilities remain balanced and there is no need to condition.

The SEP can be described in an analogous fashion. There are N players; each is given an envelope containing a *fixed* amount x, allowed to see the contents, and has the option of sticking with this amount or having the amount either doubled or halved on the basis of a coin toss. The players cannot communicate during the game: what strategy should they adopt to maximize the total amount? We note that this problem is sometimes referred as the double or halves game.

The basic logic of the subsequent discussion holds for both the TEP and the TEP+, but the details vary slightly because in the TEP y can be considered a random variable, while in the TEP+, y is a constant real positive number.

In order to clarify our subsequent discussion, we will continue to use the notation y when discussing the TEP or TEP+ and the notation x for the SEP. The major source of confusion when dealing with the TEP or TEP+ is assuming the equally likely outcomes on switching envelope are $\{y/2, 2y\}$. In fact, as the multiple player version of the game (TEP+) makes quite obvious, the outcomes can only be $\{y, 2y\}$ (these are fixed quantities) and when the player opens the first envelope, he gains no information (and does not know whether he holds y or $2y$). Nevertheless, the logic generally followed is to assume the player holds the amount y and that the other envelope contains either $y/2$ or $2y$. The computation for the expected value on changing envelope is then:

$$\frac{1}{2}\left(\frac{y}{2} + 2y\right) = \frac{5y}{4} > y, \tag{1}$$

so, apparently, the player, who already holds y, should always change envelope to obtain $5y/4$ on average.

The conundrum is that common sense or intuition suggests there should be no advantage in switching envelopes. In fact, while (1) is the correct computation for the SEP, it is incorrect for the TEP+ (or the TEP). While we will justify this more carefully in the next section, consider the following: suppose the player picks an envelope in the TEP, opens it and calls the amount A. Then either $A = y$ in which case the other envelope contains $2y$ or $A = 2y$ in which case the other envelope contains y. In either case, on switching envelope, the sample space of equally likely outcomes is $\{y, 2y\}$, *not* $\{y/2, 2y\}$. Note that in the TEP+, there are precisely two *fixed* amounts $\{y, 2y\}$ in all $2N$ envelopes. It makes no sense to perform computations of expected values based on the amount $y/2$! (in Section 7, we will support this statement in a formal way).

For the SEP, when the player opens the envelope, let the amount it contains be x. He now tosses a coin and based on the result, the amount is either doubled to $2x$ or halved to $x/2$. In this case, the sample space *is* $\{x/2, 2x\}$ and the expected value obtained on switching is

$$\frac{1}{2}\left(\frac{x}{2} + 2x\right) = \frac{5x}{4} > x. \tag{2}$$

Therefore, the player makes a gain by switching ($5x/4 > x$), on average, and the correct strategy is to always switch, not stick. The source of confusion in the TEP is to confuse it with the SEP.

3. Equivalence of sticking or changing envelope for the TEP+: two elementary proofs

There have been many attempts to put the paradox to bed. We present here two elementary proofs that always changing envelopes in the TEP+ gives no gain or advantage.

Proof 1

Each player reasons that the two envelopes contain the *fixed* amounts $y, 2y$. When the first envelope has been opened, the player does not know which amount it contains (he does not know what y is) and in fact opening the envelope provides no useful information. However, the player can reason as follows: either the opened envelope contains y or $2y$. With probability 1/2, it contains y and switching envelopes means the player gives back y and receives $2y$. Relative to his current position, this leads to a gain of

$$G_1 = -y + 2y = y.$$

Also with probability 1/2, the opened envelope contains $2y$ and switching envelopes (the player gives back $2y$ and receives y) leads to a gain of:

$$G_2 = -2y + y = -y.$$

As both of the above possibilities occur with probability 1/2, the expected value of the gain on switching envelopes is thus

$$\frac{1}{2}(G_1 + G_2) = \frac{1}{2}(y - y) = 0,$$

and there is no advantage in always changing envelopes, in the long run.

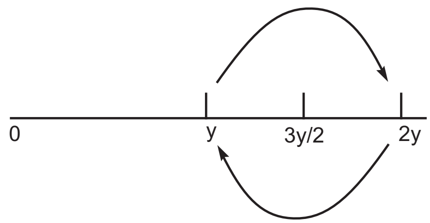

FIG. 1. The TEP. The player starts at either y or $2y$. If he chooses to change envelope, he then hops to the other value. There is no advantage for the player in switching envelope on average.

Proof 2

The two envelopes contain the fixed amounts y, $2y$. With the knowledge that opening the first envelope provides no useful information, one can proceed as follows: before the envelope is opened, the player reasons that on average (over a large number of trials), the amount each player will receive if he accepts the contents of envelope one is $\frac{1}{2}(y+2y)=\frac{3}{2}y$.

He then opens the envelope. He now reasons that changing envelopes will again yield the same average $\frac{3}{2}y$. Hence, there is no advantage in changing.

4. The source of the conundrum: the pictorial approach

It is remarkable that the TEP still causes confusion and provokes discussion. Although the simple arguments in the previous section seem obvious, they do not perhaps explain completely the source of the confusion. In our opinion, the following simple pictures clarify the issue.

We consider the TEP or TEP+ where the amounts in a pair of envelopes are $\{y, 2y\}$ and the SEP where the amount in the envelope is x and the player can choose to change envelope whence the amount becomes $x/2$ or $2x$ with equal probability, i.e. via a coin toss. In each case, the player wishes to choose a strategy to maximize his winnings on average (i.e. over a large number of repetitions of the experiment). It is important to note that the final profit is always estimated relative to the starting point, i.e. the contents of the first envelope in each case. In the TEP, the starting point is moveable in the sense that the player does not know if the amount in the first envelope is y or $2y$. To obtain a gain, the player needs to improve on the starting point. However, in the SEP, the starting point is the fixed amount x and he may step in either direction (towards $x/2$ or $2x$) depending on the coin toss.

Figure 1 refers to the TEP or TEP+ and we see that on opening envelope one, the player is either at y or $2\ y$. If he changes envelope, he stands to lose or win an equal amount, i.e. he takes a fixed step in either direction depending on his starting point. It is clear that over a large number of repetitions, given that the player starts on each end point with equal probability, that his winnings oscillate about the centre of gravity $3y/2$ and on average he neither gains nor loses by changing envelope.

Now consider Fig. 2 which refers to the SEP, we see that the player starts at x. If he chooses to change envelope, he may jump in either direction depending on the coin toss, but the jumps are not symmetrical. In fact, in a manner of speaking, by switching envelopes he stands to lose a little but gain a lot! (This is a biased random hop, starting from a fixed point, with the bias in the positive direction.) It is clear that on average over a large number of repetitions, he will gain by changing envelopes. If one were to stand on x and arbitrarily hop either left or right, over a large number of repetitions one would converge to the point $\frac{1}{2}(\frac{x}{2}+2x)=\frac{5x}{4}$. Always switching is the correct strategy for the SEP.

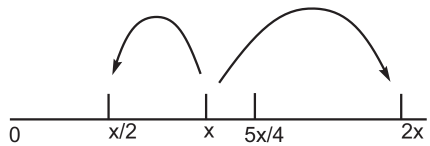

FIG. 2. The SEP. The player starts at x. If he chooses to change envelope he then hops, with equal probability, to either $x/2$ or $2x$. This is, in fact, a positively biased random hop. It is advantageous for the player to switch envelope on average.

5. Numerical simulations

It is interesting that even in numerically simulating the problem, one must be careful. A simple trap is the following: in order to simulate the TEP, one would like to numerically generate a large number of pairs of real numbers $y, 2y$. Apparently, one way to do this is to first generate a random string of real numbers (y values), using a random number generator. Then produce the companion of each via a 'coin toss', either doubling or halving the y value. A little reflection suggests that this might in fact become a simulation of the SEP, not the TEP, unless we are careful.

5.1 Simulation of TEP+

A simple way to simulate the TEP+ goes as follows (which also helps to make clear that there is no conundrum). Randomly generate a sequence of N **ordered** pairs of numbers $\{1, 2\}$ or $\{2, 1\}$ stacked beside each other in 2 rows or N columns so that we form an $2 \times N$ matrix $\mathbf{A}$, e.g.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 1 & 1 & 2 & \dots \\ 2 & 1 & 1 & 2 & 2 & 1 & \dots \end{pmatrix}$$

(Each entry in $\mathbf{A}$ is the contents of an envelope as perceived by each of the N players, each column represents the contents of the two envelopes for one player so there are $2N$ envelopes in total). The random ordering is important. Now find the average of each row (i.e. average each row over the number of players N). If sticking and changing envelope are equivalent (i.e. if there is no advantage to changing row (envelope)), then the two averages should be the same. Figure 3 simulates a repetition of the N player game carried out M times (to obtain M averages). As, on average, there must be as many $1's$ as $2's$ in each row, it is clear that each row must average to $(1 + 2)/2 = 1.5$, as is demonstrated in the simulation. The left plot gives the averages obtained from the first row (i.e. by sticking with envelope one) and the right plot gives the averages by switching to envelope two. The average of all these simulations is depicted in each as the thick solid line. (As y is a constant in the TEP+, the actual amounts in the two envelopes are obtained by multiplying $\mathbf{A}$ by y.)

5.2 Simulation of TEP

In this version, one considers y to be a random variable (see Section 2). One standard 'explanation' for the two envelope conundrum (discussed in the next section) argues that one cannot form a uniform distribution on the real numbers so that the calculation (1) becomes incorrect for this reason (the comment about the uniform distribution is technically true, but misleading from the point of view of comparing the strategies of always sticking and always changing). In actual fact, the distribution of numbers is irrelevant. To illustrate this note that we could modify the simulation above by multiplying

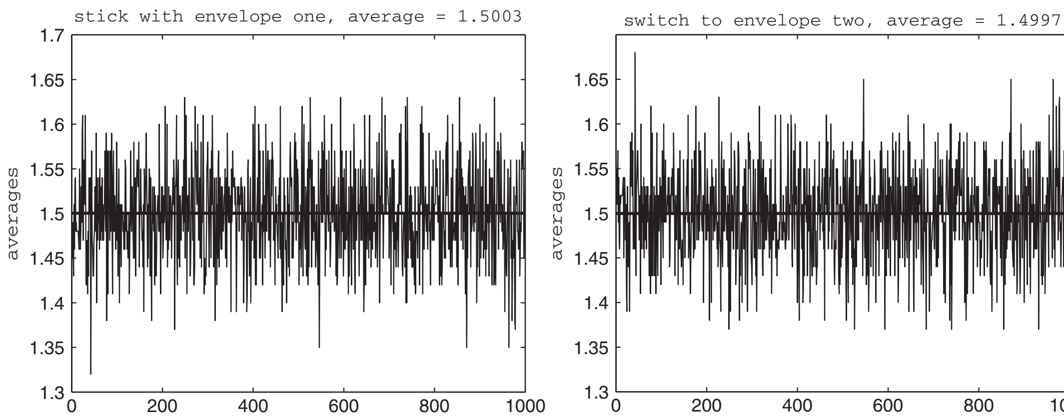

FIG. 3. The TEP: the left plot shows the averages if the player sticks with envelope one and the right plot shows the averages if the player switches to envelope two. The parameter values are $N=100$, $M=1000$.

each column of the matrix $\mathbf{A}$ by a 'random' positive real number y_j (uniformly pseudo-random insofar as it is generated by Matlab's random number function), $(j = 1, \dots, N)$ in order to create a set of envelopes. We denote this matrix by $\mathbf{B}$. Column j of the matrix $\mathbf{B}$ therefore contains either y_j, $2y_j$ or $2y_j$, y_j, e.g.

$$\mathbf{B} = \begin{pmatrix} y_1 & 2y_2 & 2y_3 & y_4 & y_5 & 2y_6 & \dots \\ 2y_1 & y_2 & y_3 & 2y_4 & 2y_5 & y_6 & \dots \end{pmatrix}.$$

It is clear that averaging across any row the average element is $1.5E(y)$, where $E(y)$ is the expected value of y as determined by the distribution from which each y_j is taken. More precisely, the expected value of any element is

$$E(B_{ij}) = E(A_{ij}y_j) = E(A_{ij})E(y_j) = 1.5E(y).$$

Of course, $E(y)$ would not exist if the y_j were genuinely uniformly distributed on the real line. But this is not the reason that (1) is fundamentally the wrong calculation. As Matlab generates its pseudo-random numbers from a well-defined distribution, $E(y)$ does exist for the simulation, but (1) is still the wrong calculation. And a little reflection suggests the underlying distribution of y_j does not change the evaluation of the strategies of always sticking and always changing envelope. To illustrate this, in Fig. 4 we generate the y values as follows: we sample x from a normal distribution with mean $\mu = 4$ and variance $\sigma^2 = 1$ and we then square this value $(y = x^2)$ to ensure we obtain a positive real number with expected value:

$$E(y) = \int_{-\infty}^{\infty} x^2 \frac{1}{2\pi\sigma^2} e^{-(\frac{(x-\mu)^2}{2\sigma^2})} dx = \mu^2 + \sigma^2.$$

We conclude that there is no advantage in choosing one row of $\mathbf{B}$ over the other (sticking or changing are equivalent) and the distribution from which y_j is chosen does not influence this fact.

5.3 *Simulation of SEP*

We also include a simulation of the SEP (with the random variable x factored out if we are playing the single player version). We use a coin tossing algorithm to create a vector (size N) consisting

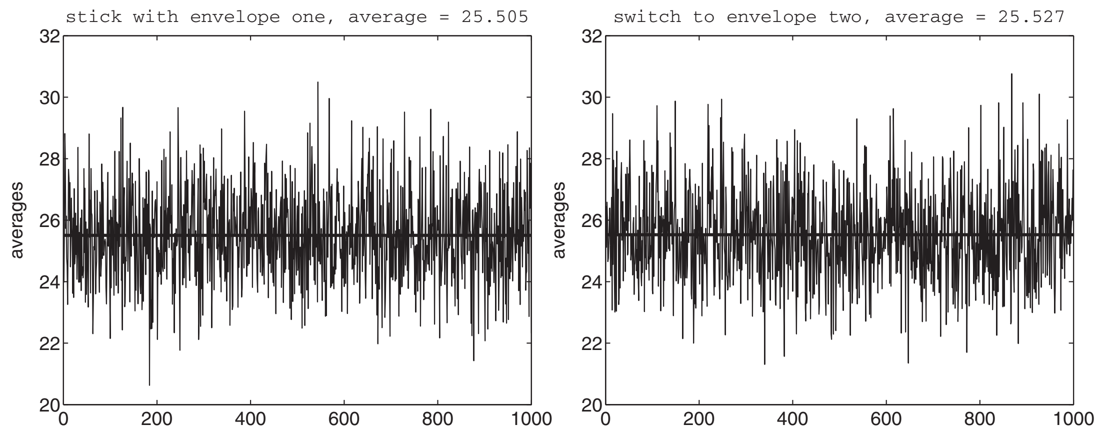

FIG. 4. The TEP: the left plot shows the averages if the player sticks with envelope one and the right plot shows the averages if the player switches to envelope two. The parameter values are $N = 100$, $M = 1000$. y, (with $E(y) = 17$) is generated by taking the square of a number drawn from a normal distribution (with mean 4 and variance 1). Both the plots appear to fluctuate around the value $1.5E(y) = 25.5$.

of a random series of $1's$ and $0's$ (with the $1's$ corresponding to heads say, and $0's$ corresponding to tails):

$$F = [1, 0, 1, 1, 0, 0, 0, 1, \ldots].$$

Suppose the player starts off with an amount 1. Assuming they always switch, we assign the value 0.5 if the i-th entry for F is 0 and 2 if the i-th entry is 1. This creates a vector

$$\mathbf{v} = [2, 0.5, 2, 2, 0.5, 0.5, 0.5, 2, \ldots].$$

Then the average amount won in N switches is W, where

$$W = \frac{1}{N}\sum_{j=1}^{N} v_j. \tag{3}$$

In an identical manner to the TEP simulations, we then repeat the experiment M times to give W_i for $i = 1, \ldots, M$, where W_i are defined in (3) for each experiment. Also set

$$\overline{W} = \frac{1}{M}\sum_{i=1}^{M} W_i.$$

Then, Fig. 5 shows all the W_i and $\overline{W}$ (given as a thick solid line). We see that this is converging to $5/4 = 1.25$, as expected. This is the advantage ($1.25 > 1$) one gains by switching in the SEP.

6. The *J* envelope game

The TEP is generalizable to a situation with $J \geq 2$ envelopes, where a number of these $S \leq J$ contains the *fixed* amount y, the smaller amount, and the remainder $J - S$ contain the larger *fixed* amount $2y$. Each of the N players (who cannot communicate during the playing of the game) is given J envelopes, and knows that S contain y and the remainder $J - S$ contain $2y$. He is allowed to open any one of the

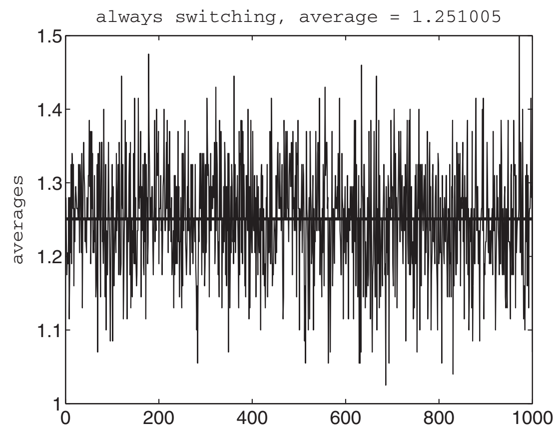

FIG. 5. The SEP: the plot shows the averages if the players always switch envelope. The parameter values are $N = 100$, $M = 1000$.

J envelopes. He is then given the option of sticking or switching. Intuition suggests that there should be no advantage in switching (or sticking). It is interesting to use this problem as a consistency check. Following the methodology of Proof 1 in Section 3, let us check this carefully. One reasons that when a player opens an envelope:

- he holds amount y with probability $p_1 = S/J$. If he switches envelope he will (on average) stand to gain:

$$G_1 = -y + \frac{S-1}{J-1}y + \frac{J-S}{J-1}2y \quad \text{with probability } p_1 = \frac{S}{J},$$

- he holds amount $2y$ with probability $p_2 = (J-S)/J$. If he switches envelope he will on average stand to gain:

$$G_2 = -2y + \frac{S}{J-1}y + \frac{J-S-1}{J-1}2y \quad \text{with probability } p_2 = \frac{J-S}{J}.$$

Thus, one reasons that if the player switches, the expected value can be computed directly and we find that regardless of the value of y:

$$p_1 G_1 + p_2 G_2 \equiv 0,$$

and there is no advantage in always switching in agreement with the intuitive answer. Note that when $J = 2$, $S = 1$, this game reduces to the TEP+ as a special case.

7. A formal approach using random variables

Because previous formal approaches have apparently not resolved the conundrum, the emphasis up to this point has been on an informal approach to the problem in an attempt to understand the source of the confusion. Having made our point, we now present a concise formal proof to support what has come before. The formal approach has the disadvantage that it does not perhaps make clear why this problem causes confusion.

We will first describe the random experiment. One of the sources of confusion in this problem is the 'incomplete knowledge' which a player would have on opening a single envelope. Instead of playing the 'game', we propose the following experiment: there are $i = 1 \ldots N$ pairs of envelope each pair containing the amounts $\{y_i, 2y_i\}$, where y_i is the smaller amount. Of each pair of envelopes, one is chosen via a coin toss and opened, the amount observed is noted, and then the second envelope is also opened and the amount in it is noted. In this way, we have complete knowledge of the contents of the envelopes (which a player of the 'game' would not). However, using the complete knowledge gained from this experiment, we can estimate whether or not there is any advantage in the strategy of sticking with the original envelope or switching to the second, i.e. we can estimate the expected value of what a player would obtain by either always sticking or always swapping.

A possible sample space for the experiment is

$$U = \{(u_1, u_2), u_1 \in \{1, 2\}, u_2 \in \mathbb{R}^+\}, \tag{4}$$

where u_1 is an auxiliary variable.

It is useful to define random variables Z, Y. We define Z via

$$Z(u_1, u_2) = u_1 \tag{5}$$

so that Z takes the value 1, respectively 2, if we choose the envelope with the lowest, respectively highest value, both with probability 1/2. We also define the random variable:

$$Y(u_1, u_2) = u_2 \tag{6}$$

to be the actual value of the smaller amount. Y, Z are clearly independent random variables. The value revealed if the player sticks with the envelope first chosen is then given by the random variable:

$$X_s = YZ. \tag{7}$$

7.1 *The correct calculation for sticking or changing envelope*

If the player chooses to keep the first envelope chosen the expected value is:

$$E(X_s) = E(YZ) = E(Y)E(Z) = 1.5E(Y), \; Y, Z \text{ are independent.} \tag{8}$$

If the player chooses to change envelope, then the associated random variable for this alternative strategy is

$$X_c = Y(3 - Z) \Rightarrow E(X_c) = E(3 - Z)E(Y) = 1.5E(Y), \; Y, Z \text{ are independent.} \tag{9}$$

As $E(X_s) = E(X_c)$, there is no advantage in changing envelope.

7.2 *The conundrum: the incorrect calculation*

The conundrum arises because of the following incorrect reasoning based on the switching envelope strategy. We first note the standard result:

$$E(X_c) = \sum_{z=1,2} E(X_c|Z = z)P(Z = z) = E(X_c|Z = 1)P(Z = 1) + E(X_c|Z = 2)P(Z = 2). \tag{10}$$

But recalling that $X_s = YZ$ and $X_c = (3 - Z)Y$ we have

$$X_c|(Z = 1) = 2Y = 2ZY|(Z = 1) \text{ and } X_c|(Z = 2) = Y = YZ/2|(Z = 2) \tag{11}$$

so that

$$\begin{aligned} E(X_c) &= E(2YZ|(Z = 1))P(Z + 1) + E(YZ/2|(Z = 2))P(Z = 2) \\ &= E(2X_s|(Z = 1))P(Z = 1) + E(X_s/2|(Z = 2))P(Z = 2). \end{aligned} \tag{12}$$

The conundrum follows if we make the incorrect assumption that Z and $X_s = YZ$ are independent which leads to:

$$E(2X_s|Z = 1) = E(2X_s) \text{ and } E(X_s/2|Z = 2) = E(X_s/2) \tag{13}$$

so that we obtain the result:

$$E(X_c) = 1/2E(2X_s) + 1/2E(X_s/2) = 5/4E(X_s) > E(X_s) \tag{14}$$

and apparently, but incorrectly, switching envelopes is advantageous.

8. Conclusions

We have presented two simple arguments, supported by numerical simulations, to resolve the apparent conundrum of the TEP. We show that the intuitive answer, that there is no advantage in always switching envelope, is in fact correct and there is no conundrum. Figures 1 and 2 illustrate pictorially what we perceive to be the source of the confusion. We have also introduced a generalized version of the problem where there are $J \geq 2$ envelopes and we have shown that the intuitive answer (no advantage in always switching) is also correct in this case. Finally, we support the informal approach with a more formal argument in Section 7.

Perhaps, the simplest way to present the TEP in the future is as the multiple player version (TEP+) as described in Section 2. We have shown above that the TEP+ formulation of the problem removes the 'uniform distribution' complication as the amounts in all $2N$ envelopes are the constants y and $2y$, i.e. y is a constant, whose value is not known to the players.

If one nevertheless wishes to use the single player version (TEP), where y must be taken to be a random variable rather than a constant, we suggest the following modification of the description of the game: a player is presented with N pairs of envelopes, one holding twice as much money as the other *where the lower amount is drawn from a well-defined distribution on* $\mathbb{R}^+$ which is not revealed to the player. Of each pair of envelopes, the player is allowed to open one envelope and, having seen the contents, may either keep that amount or opt to change envelope. Are the strategies of always sticking and always changing envelope equivalent to obtain the maximum amount of money over many repetitions? Why is it incorrect to use the calculation of (1) which suggests switching envelope is advantageous?

In our opinion, there are many over elaborate 'resolutions' of the conundrum, some of which merely add to the confusion. A typical example is Devlin (2004). It is often claimed, for example, that the confusion in the TEP arises because one cannot form a uniform distribution on the real line (Blachman *et al.*, 1996; Devlin, 2004) (or that one is dealing with infinite quantities) so the calculation (1) is wrong for this reason. While we believe that the TEP+ version is the simplest to explain, if one nevertheless wishes to discuss the single player version of the game (TEP), we showed in Section 5 why the distribution argument is not the source of the 'conundrum'. The distribution from which y is drawn is irrelevant. (In Section 5, different distributions were discussed.) In any case, if this were the origin of the difficulty, would this argument not also hold for the SEP? Yet the solution of the SEP is not in dispute. Consider a version of the SEP where the amount in the envelope comes from the following non-uniform (discontinuous) probability distribution:

$$f(x) = \begin{cases} \dfrac{1}{4}, & 0 < x < 1 \\ \dfrac{3}{4x^2}, & 1 \leq x < \infty. \end{cases}$$

In this case, one can verify directly that $E(x) = 13/8$, but the player does not know this. The calculation of the expected value on switching turns out to be the same as before, i.e. $5E(x)/4$ in the long run, and one should always switch. (In this case, sticking with the envelope will lead to an overall average of $E(x) = 13/8$ while switching will lead to $5E(x)/4 = 65/32$.)

In a classroom setting, we suggest that this problem could be introduced by running an experiment where the class plays the TEP+ game attempting to maximize profits. Half the class could use the switching strategy, the other half could stick and the results could be compared. The results could be compared with a similar experiment playing the SEP. Of course, one can also devise more complicated versions of TEP+, where players who believe that switching yields an advantage along the lines of equation (1) are asked to pay a penalty to switch envelope (which is less than the supposed $y/4$ advantage). If the first exposure of a particular group to this problem was empirical, and the results were as reported in this article, it is tempting to suggest that such a group would realize *ab initio* that there is no conundrum. This would make an interesting study.

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Appendix

The Monty Hall problem (Shaughnessy & Dick, 1991; Rosenhouse, 2009) can be represented as a game in the following way: three envelopes are placed in front of the player by the game host who knows the content of each envelope. Two of the envelopes contain a single Euro note, and the other contains a 1 million Euro note. The player is asked to select one envelope. This is *not* opened and put aside. The host then opens one of the two remaining envelopes (at least one must contain a single Euro note) and shows the player that it contains 1 Euro. The player is then given the choice of sticking with the original envelope (still unopened) or switching to the other unopened envelope. Is there any advantage in changing envelope?

It is not our intent to discuss the problem at length here. After some controversy, it has been well established that the opening of the envelope containing 1 Euro provides extra information and it is advantageous for the player to switch envelope (Rosenhouse, 2009). The probability of success is 1/3 if the player sticks and 2/3 if the player changes. Many people (including senior statisticians!), on first being introduced to the problem, believe instinctively that there is no advantage in changing envelope.

As an introduction to this problem suitable for the classroom we suggest the following approach. We wish to demonstrate that changing envelope *does* confer some sort of an advantage (i.e. gives extra useful information). Consider an exaggerated version of the game where there are 1,000,000 envelopes, precisely one of which contains 1 million dollars. As before, the player is asked to select one envelope which is temporarily put aside. (The chances of this being the correct envelope are indisputably 1/1,000,000.) The host now opens 999,998 of the remaining envelopes, all of which are shown to contain 1 Euro. At this point, there are only two envelopes left: the originally selected unopened envelope and one more. One of these two envelopes definitely contains the 1 million Euro note. The player is asked if they want to stick with the original envelope or change. It is clear that the chances of the player having originally selected the correct envelope remain at 1/1,000,000 so the chances that the second envelope contains the 1 million Euro note must be 999,999/1,000,000. So is there an advantage in changing envelope at this point? The game can also be formulated in terms of lottery numbers. To finish, a classroom demonstration might involve having an envelope for each member of the class except the player.

S. L. Mitchell is a lecturer in the Department of Mathematics and Statistics, currently Deputy Head of Department, at the University of Limerick. Her research interests include numerical and mathematical modelling of moving boundary problems relevant to solidification, melting and polymer diffusion, and modelling hydraulic fractures in rocks. She is also currently an Associate Editor for the Applied Mathematics and Computation Journal.

S. B. G. O'Brien is a Professor in the Department of Mathematics and Statistics at the University of Limerick. His research interests include mathematical modelling, industrial mathematics, thin film flows and asymptotic techniques. In 2006 he founded the Mathematics Applications Consortium for Science and Industry (MACSI), which is supported by Science Foundation Ireland (SFI), and consists of a network of mathematical modellers and scientific computational analysts involved in mathematical modelling and industrial mathematics. He is also currently President of the European Consortium for Mathematics in Industry (ECMI).