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Improving the accuracy of heat balance integral methods applied to thermal problems with time dependent boundary conditions

S.L. Mitchell [a,*], T.G. Myers [b]

[a] MACSI, Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland
[b] Centre de Recerca Matemàtica, 08193 Bellaterra, Barcelona, Spain

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ABSTRACT

In this paper the two main drawbacks of the heat balance integral methods are examined. Firstly we investigate the choice of approximating function. For a standard polynomial form it is shown that combining the heat balance and refined integral methods to determine the power of the highest order term will either lead to the same, or more often, greatly improved accuracy on standard methods. Secondly we examine thermal problems with a time-dependent boundary condition. In doing so we develop a logarithmic approximating function. This new function allows us to model moving peaks in the temperature profile, a feature that previous heat balance methods cannot capture. If the boundary temperature varies so that at some time $t > 0$ it equals the far-field temperature, then standard methods predict that the temperature is everywhere at this constant value. The new method predicts the correct behaviour. It is also shown that this function provides even more accurate results, when coupled with the new CIM, than the polynomial profile. Analysis primarily focuses on a specified constant boundary temperature and is then extended to constant flux, Newton cooling and time dependent boundary conditions.

1. Introduction

Consider the problem of heating a semi-infinite solid by specifying a time dependent temperature at the boundary $x = 0$. In non-dimensional form this may be specified as

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < \infty, \tag{1}$$

$$T = 0, \quad \text{at } t = 0, \tag{2}$$

$$T = h(t), \quad \text{at } x = 0, \tag{3}$$

$$\frac{\partial T}{\partial x} \to 0, \quad \text{as } x \to \infty. \tag{4}$$

Using Green's functions (or Laplace transforms and the convolution theorem) the solution for any $h(t)$ can be written in integral form as

$$T(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{h(\tau) e^{-x^2/4(t-\tau)}}{(t-\tau)^{3/2}} d\tau = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{h(t-\tau) e^{-x^2/4\tau}}{\tau^{3/2}} d\tau. \tag{5}$$

Of course it might not be possible to explicitly evaluate this integral for all h. Hence approximate solution methods may also be necessary. Furthermore, if we see the above thermal problem as a starting point for a more complex Stefan problem, then approximate solutions are required except in a small number of special cases.

One suitable approximate technique is the well-known Heat balance integral method (HBIM), developed by Goodman [7,8]. For the above initial-boundary value problem the HBIM involves three steps:

(a) First we define the heat penetration depth, $\delta(t)$. For $x \geqslant \delta$ the temperature change from the initial temperature is negligible.
(b) An approximating function, typically a polynomial, is then introduced. This describes the temperature for $0 \leqslant x \leqslant \delta(t)$.
(c) Finally, the heat equation is integrated over $x \in [0, \delta]$ to produce the heat balance integral. This results in a single ordinary differential equation for δ, which may often be solved analytically.

Once δ has been determined, the temperature T is known from the approximating function which is given in terms of δ. An alternative approach, developed by Sadoun and Si-Ahmed [26], is the Refined integral method (RIM), which simply involves carrying out a double integration of the heat equation at stage (c). The relative merits of the two approaches, as well as a number of variations involving alternative approximating functions is described in detail in [17]. In fact, this method has appeared at least twice before in the literature. Gupta and Banik [9] developed what they

* Corresponding author.
E-mail addresses: sarah.mitchell@ul.ie (S.L. Mitchell), tim.myers@ymail.com (T.G. Myers).

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termed as the *constrained integral method* for solving moving boundary problems. Their method involves applying what is termed as the zeroth and first moments to the approximating polynomial: these are precisely the HBIM and RIM formulations, respectively. Also, Hill [10] described the RIM formulation but referred to it as the *integration formulation by integration*.

Both the HBIM and RIM have two well-known drawbacks [17,20,21]. Firstly, the accuracy depends on the choice of approximating function [17]. This appears to be problem dependent and consequently there exist a wide variety of options within the literature [11,17,28]. Quadratic, cubic, quartic, non-integer powers and exponential approximating functions have all been tried, with varying degrees of success [1,4,8,16,19,23,24]. Ref. [28] focusses on the accuracy for different formulations of the quadratic problem. When dealing with Stefan problems, without an analytical solution, the work in [17] compares different formulations of both quadratic and cubic approximations, while [15] compares the accuracy of a quadratic HBIM against perturbation and numerical solutions. Secondly, when applying a time-dependent boundary condition, heat balance methods are only possible for small time or very limited (and not physically realistic) forms of $h(t)$, see [4,14,18,25,27] for example.

The first issue, namely the choice of approximating function, was recently addressed for thermal problems in [20] and subsequently applied to Stefan problems in [21]. With the correct scaling the temperature can be approximated by a polynomial form $T \sim (1 - x/\delta)^n$, where n is most often set to 2 or 3. The approach adopted in [20,21] was to leave n unknown. A further condition must then be introduced to determine n. Langford [13] defined an error for the HBIM problem. In [20,21] n is chosen to minimise this error. This method is discussed in more detail in Section 2.1. The solutions obtained using this minimisation method are more accurate than with the standard method, often significantly so. However, the algebra can be difficult and so the optimal values of n for constant temperature, constant flux and Newton cooling boundary conditions are tabulated in [20,21]. The method we propose in the following involves combining the HBIM and RIM solutions to determine the exponent, which we refer to as the *combined integral method* (CIM). This approach is much simpler to apply than the minimisation technique.

We also tackle the second issue, that of applying time-dependent boundary conditions. Previous authors have used such conditions with a standard HBIM formulation applied to Stefan problems. Generally the boundary conditions are not physically realistic, *e.g.* of the form $h(t) = t^m$, e^t [4,18,25] or results are presented for small times (before the analytical solution fails) [6,12,14,27]. It was also noted by Goodman [7] that for Stefan problems with a time dependent heat flux, the HBI method is only useful for functions which are monotonically increasing or constant. When applying the minimisation technique to time-dependent boundary conditions (where n may vary with time) the algebra can be prohibitively difficult. Even for a cooling condition the problem that $n = n(t)$ arises. In [20,21] this is avoided by fixing $n(t) = n(0)$, but the accuracy of results is then reduced. With the new CIM the algebra remains tractable and no such approximation is necessary. The time-dependent boundary condition can also lead to a catastrophic failure in the integral methods. For example, if we choose $h(t) = 1 - t$, with a far-field temperature $T = 0$, then at $t = 1$, when $h(t) = 0$, both the HBIM and RIM predict an infinite heat penetration depth and $T = 0$ everywhere. To deal with this we introduce a new form of approximating function. When used with the CIM the new form turns out to be significantly more accurate than the polynomial profiles.

Bell [2,3] has suggested an alternative way to improve the accuracy of Goodman's basic method by developing a piecewise linear heat balance integral solution. In fact this is a form of finite difference numerical scheme that uses the HBIM on each subinterval. It has first order convergence whereas more standard finite difference approaches have better convergence, such as the second order accurate method of Mitchell and Vynnycky [18].

We begin the paper by analysing the problem for constant $h(t) = 1$ in order to explain the new method of determining the approximating function. Subsequently, we investigate different forms of boundary condition and specifically problems where $h(t)$ varies with time in such a way that the standard approximations break down.

2. Standard thermal boundary conditions

The following sections deal with developing the new method for constant temperature, constant flux and Newton cooling boundary conditions. We then pose the question, is this method an improvement on previous methods? Finally we investigate whether the exponent n can vary with time.

2.1. Constant temperature boundary condition

Taking $h(t) = 1$ in (5) and applying (2) gives the well known solution

$$T(x,t) = \operatorname{erfc}\frac{x}{2\sqrt{t}}. \tag{6}$$

The standard approximating polynomial for the HBIM has the form

$$T = \left(1 - \frac{x}{\delta}\right)^n. \tag{7}$$

This satisfies the far-field condition $T_x(\delta,t) = 0$ provided $n > 1$. In addition, a further condition on the curvature is often applied. It is derived by noting

$$\frac{DT}{Dt}(\delta,t) = \left(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{d\delta}{dt}\right)\bigg|_{x=\delta} = 0 \quad \Rightarrow \quad \frac{\partial^2 T}{\partial x^2}(\delta,t) = 0. \tag{8}$$

If this condition is also imposed then we require $n > 2$. In general n is chosen arbitrarily, and usually to provide the simplest profile that satisfies the boundary conditions. Consequently $n = 2$ or 3 are the most common choices. With n specified the problem reduces to determining δ.

In an attempt to improve the accuracy of the HBIM solution Myers [20] left n as an unknown and determined it as part of the solution. The profile (7) therefore involves two unknowns, the heat penetration depth $\delta(t)$ and the exponent n (initially assumed constant). The heat penetration depth is determined via the heat balance integral, which may be written in the form

$$\int_0^\delta \left[\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2}\right] dx = 0. \tag{9}$$

This leads to a first order differential equation for δ. From Eq. (9) we can see the weakness of the HBIM, namely that the integral does not require T_t to closely approximate T_{xx} merely that the area beneath $f(x,t) = T_t - T_{xx}$ is zero. Perhaps the worst case scenario is when f is odd about $\delta/2$ and can then be arbitrarily large elsewhere. In practice the difference is usually quite small but this argument does highlight why some choices of approximating function are better than others. To avoid this problem Myers [20] also requires that n is chosen to minimise the error defined by Langford [13]

$$E_n = \int_0^\delta [f(x,t)]^2 dx = \int_0^\delta \left[\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2}\right]^2 dx \geqslant 0. \tag{10}$$

Minimising the square of $f(x,t)$ prevents the cancelling of positive and negative areas inherent in the HBIM and consequently forces the terms T_t and T_{xx} to closely approximate each other. For a constant temperature boundary condition it was shown in [20] that the error is minimised by choosing $n = 2.2335$. This obviously lies between 2 and 3, so clarifying the preference for choosing these integer values.

This minimisation technique will provide results that are significantly more accurate than the standard HBIM approach and is particularly useful for Stefan problems, for which there are few analytical solutions. However, one disadvantage of this new method is that the integral (10) can be quite complex. For the three standard thermal boundary conditions, namely constant temperature, constant flux and Newton cooling, the appropriate values of n are tabulated in [20] and are quoted here later on in this section for convenience. However, for different scenarios such as those encountered in Stefan problems the integral may need to be calculated numerically. A second issue occurs with certain types of boundary condition. For example, when applying a Newton cooling condition, $T_x = T - 1$ at $x = 0$, the value of n that minimises the error changes with time. If n is defined as $n \equiv n(t)$ then the integral E_n becomes even more complicated due to the T_t term. This difficulty may be circumvented by noting that for this boundary condition the error is greatest when $t = 0$ and so n may set to the constant value $n(0)$. Since $T = 0$ at $t = 0$ the boundary condition $T_x = T - 1$ reduces to the constant flux condition $T_x = -1$, which then determines the value of n.

We now offer an alternative approach to obtain n which is less algebraically complex than the minimisation technique. As in the method of [20] we sometimes find $n = n(t)$, but this is more easily dealt with due to the simpler form of the governing equations and consequently we do not need to set n as a constant. First we write the HBIM (9) in a more standard form, by integrating the second term (noting $T_x(\delta, t) = 0$) and taking the time derivative outside of the integral (noting $T(\delta, t) = 0$)

$$\frac{d}{dt}\int_0^{\delta} T\,dx = -\left.\frac{\partial T}{\partial x}\right|_{x=0}. \tag{11}$$

The RIM is derived by integrating the heat equation twice with respect to x

$$\int_0^{\delta}\left(\int_0^{x}\frac{\partial T}{\partial t}\,d\xi\right)dx = \int_0^{\delta}\left(\frac{\partial T}{\partial x} - \left.\frac{\partial T}{\partial x}\right|_{x=0}\right)dx.$$

Then integrating the left hand side by parts we obtain

$$\delta\int_0^{\delta}\frac{\partial T}{\partial t}\,dx - \int_0^{\delta} x\frac{\partial T}{\partial t}\,dx = -T|_{x=0} - \delta\left.\frac{\partial T}{\partial x}\right|_{x=0}. \tag{12}$$

However, since $T_t = T_{xx}$ the first term on the left hand side can be integrated again. This yields the equation for RIM

$$\frac{d}{dt}\int_0^{\delta} xT\,dx = T(0,t). \tag{13}$$

Substituting T from Eq. (7) into (11) and (13) gives the following expressions:

$$\text{HBIM:}\quad \frac{d}{dt}\left[\frac{\delta}{n+1}\right] = \frac{n}{\delta}, \qquad \text{RIM:}\quad \frac{d}{dt}\left[\frac{\delta^2}{(n+1)(n+2)}\right] = 1. \tag{14}$$

Assuming that n is constant (see Section 2.3), in both cases integration subject to $\delta(0) = 0$ leads to $\delta = \alpha\sqrt{t}$ where

$$\text{HBIM:}\quad \alpha = \sqrt{2n(n+1)}, \qquad \text{RIM:}\quad \alpha = \sqrt{(n+1)(n+2)}. \tag{15}$$

Our proposed method requires that δ is the same for either the HBIM or RIM. In this case since the integration is trivial the simplest method is to equate the expressions for α in (15) to find $n = 2$. Alternatively we may equate the two expressions for δ_t obtained from (14), with the same result. We note here that the minimisation technique of [20] predicts $n = 2.2335$, 2.2185 for the HBIM and RIM respectively, so the value $n = 2$ is the closest integer (although there is no reason to suppose the new method leads to an integer value for n). A comparison of results for the current solution and the minimisation technique will be shown later in Section 3 once we have discussed a further modification.

2.2. Solutions for constant flux and cooling conditions

For completeness we now discuss the two other basic boundary conditions at $x = 0$. First consider the problem where the constant temperature boundary condition is changed to a constant flux condition $T_x(0, t) = -1$. The temperature takes the form $T = (\delta/n)(1 - x/\delta)^n$ and the HBIM and RIM formulations lead to the following equations:

$$\text{HBIM:}\quad \frac{d}{dt}\left[\frac{\delta^2}{n(n+1)}\right] = 1, \qquad \text{RIM:}\quad \frac{d}{dt}\left[\frac{\delta^3}{(n+1)(n+2)}\right] = \delta. \tag{16}$$

Again assuming n is constant we find $\delta = \sqrt{\alpha t}$ where $\alpha = n(n + 1)$ for the HBIM and $\alpha = 2(n + 1)(n + 2)/3$ for the RIM. As before we may either eliminate δ_t between the two equations in (16) or equate α expressions, giving $n = 4$. The minimisation technique shows that $n = 3.584$, 3.822 for HBIM and RIM. So again the new method leads to the closest integer value.

In the case of a cooling condition $T_x = T - 1$ at $x = 0$ the temperature is

$$T = (\delta/(n+\delta))\left(1 - \frac{x}{\delta}\right)^n. \tag{17}$$

Using the minimisation technique Myers [20] found that in this case n is time-dependent. If $n \equiv n(t)$ then the whole minimisation formulation requires recalculating and specifically the T_t term involves derivatives of n. The result is two simultaneous, highly complex nonlinear differential equations to determine δ and n. To avoid this it was suggested that if a constant value of n is to be used then the best choice is $n(0)$ (as the error E_n is maximum at $n = 0$). Since $T(x, 0) = 0$ this means that the optimum n is the same as that obtained via the constant flux condition, $T_x(0, t) = -1$.

With the current method, for a cooling condition the problem is governed by the following equations:

$$\text{HBIM:}\quad \frac{d}{dt}\left[\frac{\delta^2}{(n+1)(n+\delta)}\right] = \frac{n}{n+\delta}, \tag{18}$$

$$\text{RIM:}\quad \frac{d}{dt}\left[\frac{\delta^3}{(n+1)(n+2)(n+\delta)}\right] = \frac{\delta}{n+\delta}. \tag{19}$$

Assuming n to be constant and eliminating δ_t from the two differential equations we have to solve a quadratic for n, giving

$$n = \frac{4 - \delta \pm \sqrt{16 + \delta^2}}{2}. \tag{20}$$

From this it is clear that the constant n assumption is incorrect and $n = n(\delta(t)) = n(t)$. In this case the problem then reduces to the numerical solution of the two first order differential equations in (18) and (19) for $n(t)$, $\delta(t)$. This example leads to two obvious questions, namely how can we tell if n is time-dependent and if so what is the initial condition?

2.3. Is n time-dependent?

For the standard heat balance analyses n is specified as constant at the start of the calculation and δ must adjust to accommodate this. In the current method we have more freedom for n but in the first two examples above, constant temperature and constant flux, we have assumed n to be constant. This was motivated by the results of [20]. We now examine the question whether n can vary in more detail.

We begin with the constant temperature condition given by Eq. (14). Without the assumption of constant n we find

$$\frac{1}{n+1}\frac{d\delta}{dt} - \frac{\delta}{(n+1)^2}\frac{dn}{dt} = \frac{n}{\delta}, \tag{21}$$

$$\frac{2\delta}{(n+1)(n+2)}\frac{d\delta}{dt} - \frac{\delta^2(3+2n)}{(n+1)^2(n+2)^2}\frac{dn}{dt} = 1, \tag{22}$$

for the HBIM and RIM, respectively. These may be rearranged to

$$\delta\frac{d\delta}{dt} = (n+1)(4+n-n^2), \quad \delta^2\frac{dn}{dt} = (n+1)^2(4-n^2). \tag{23}$$

Obviously the second equation admits constant n solutions for $n = \pm 2, -1$. With the criteria that $n > 1$ we see that the correct choice is $n = 2$. This is the solution found from the CIM in the previous section. From (23) it follows that $\delta = \sqrt{12t}$.

Similarly, for a constant flux the HBIM and RIM formulations (16) can be written as

$$\delta\frac{d\delta}{dt} = \frac{2(n+1)(3n^2+6n+2)}{2n^2+9n+6}, \quad \delta^2\frac{dn}{dt} = \frac{n(n+1)(n+2)(4-n)}{2n^2+9n+6}. \tag{24}$$

Again the only constant n solution satisfying $n > 1$ corresponds to the solution that we quoted earlier, $n = 4$. Eq. (24) then determines $\delta = \sqrt{20t}$.

Finally, for the cooling condition Eqs. (18) and (19) become

$$\delta\frac{d\delta}{dt} = 2(n+1), \quad \delta^2\frac{dn}{dt} = (n+1)[\delta(2-n) + n(4-n)]. \tag{25}$$

Now the only possible constant n solution is $n = -1$ which violates the criteria $n > 1$ and consequently we must accept $n = n(t)$. Motivated by previous solutions we assume that as $t \to 0$ then $n \to n_0 \neq 0$ while $\delta \to At_\alpha$. Substituting this into Eq. (25) indicates $\alpha = 1/2$ and $A = \sqrt{4(n_0+1)}$, that is $\delta = \sqrt{4(n_0+1)t}$. The leading order term of Eq. (25) then gives $n_0 = 4$. This could also be deduced by noting that as $t \to 0$ the temperature $T \to 0$ and the boundary condition $T_x = T - 1$ then becomes $T_x \to -1$. So the initial condition for n must be the same as in the constant flux case and we take $\delta(0) = 0$, $n(0) = 4$.

It is worth pointing out that the method of [20] leads to a very complex expression for the error E_n when $n = n(t)$ (in fact it is a highly nonlinear integro-differential equation) and this was the main reason why the value $n(t) = n(0)$ was chosen. With the current method the governing Eq. (25), are much simpler and only involve solving two first order differential equations. In Fig. 1(a) we compare the temperatures with the exact solution, see [20], at $t = 1$ subject to a cooling condition at $x = 0$. The three curves correspond to the exact solution (solid line), the solution where $\delta(t)$ and $n(t)$ (dashed line) are calculated through (25) with $n(0) = 4$, and the solution taking the constant value $n = 4$ (dash-dotted line). It is clear that varying δ and n leads to much more accurate results. In Fig. 1(b) we show $n(t)$. From this we can see that the value $n = 4$ is only appropriate for relatively small times. If we allow $\delta \to \infty$ in Eq. (25) then the dominant term on the right hand side is $\delta(2-n)$ and so it follows that the large time asymptote is $n = 2$. Thus n will vary between 2 and 4 (further clarifying the popularity of the standard choices 2 or 3).

A close examination of the error between the exact and approximate solutions indicates that in places the CIM leads to a smaller error than that obtained when minimising the error. In the example shown in Fig. 1, due to taking constant n in the minimisation method, the CIM in fact generally provides the best approximation. This then poses the questions, is the new method an improvement on the error minimisation method and how does it produce such accurate results?

2.4. Is the new method an improvement?

To explain why a combination of HBIM and RIM may improve on the HBIM or RIM solutions alone consider the function $f(x,t) = T_t - T_{xx}$ and then define $I_1(x,t) = \int_0^x f(s,t)ds$, so the HBIM condition (9) is $I_1(\delta,t) = 0$. As discussed earlier this is not a strong condition since it does not require $T_t \approx T_{xx}$, merely that the area under $f(x,t)$ is zero. For example, any function that is odd around $x = \delta/2$ will satisfy $I_1(\delta,t) = 0$. Now define $I_2(x,t) = \int_0^x \int_0^r f(s,t)ds\,dr$; the RIM requires $I_2(\delta,t) = 0$. This integral defines a volume over a triangular base. The area of the triangle is smaller for $s \in [0,\delta/2]$ than for $s \in [\delta/2,\delta]$, hence for a function that is odd around $\delta/2$ the volume defined by the integral will be smaller for $s < \delta/2$ than the volume above $s > \delta/2$, and so the integral will be non-zero. We may see this by changing the order of integration

$$I_2(\delta,t) = \int_0^\delta \int_0^r f(s,t)ds\,dr = \int_0^\delta \int_s^\delta f(s,t)dr\,ds$$
$$= \int_0^\delta (\delta - s)f(s,t)ds = \delta I_1(\delta,t) - \int_0^\delta sf(s,t)ds. \tag{26}$$

So, if we impose the HBIM condition then the first term on the right hand side of (26) disappears and the RIM condition becomes $I_2(\delta,t) = -\int_0^\delta sf(s,t)ds = 0$. This obviously has a different form to

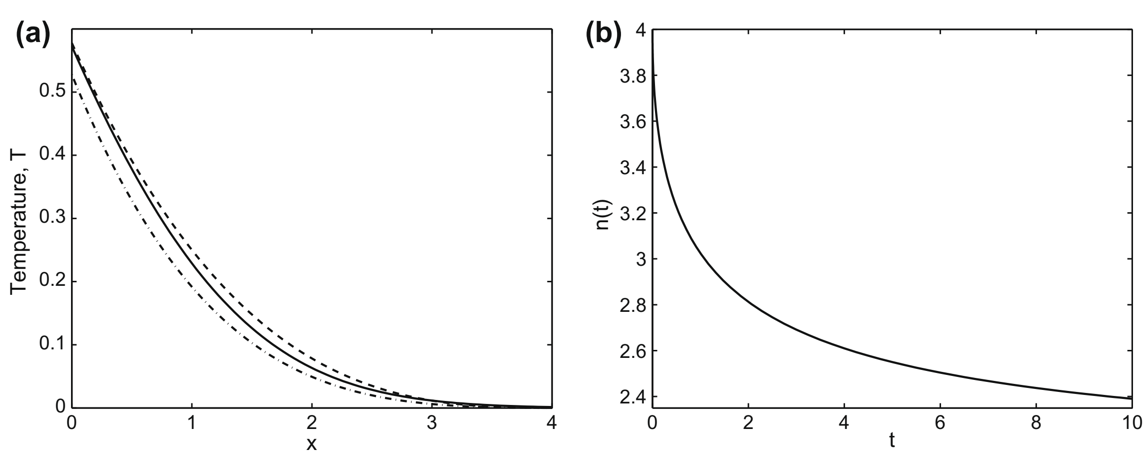

Fig. 1. Cooling boundary condition: (a) comparison of temperatures at $t = 1$ using the numerical solution of (25) to calculate δ and n (dashed line), using $n = 4$ (dot-dashed line) and the exact solution (solid line) and (b) variation of $n(t)$ up to $t = 10$.

the HBIM. In particular if f is odd around $\delta/2$ then sf is not odd and so the condition restricts f (note, since we have the odd criteria based around $s = \delta/2$ multiplying by s does not make it even). Conversely, if sf is odd around $s = \delta/2$ then f is even and I_1 restricts the size of f. Consequently, even in the worst case scenario, at least one of these conditions acts to diminish the magnitude of f. A combination of the two will either provide the same level or greater accuracy than a single condition. In other words we may expect the CIM to give errors bounded by the HBIM and RIM solutions. In the case where $n = n(t)$, due to the difficulty of implementation, the minimisation technique is used with $n = n(0)$. The CIM, which allows $n = n(t)$, is still bounded by the correct solutions, *i.e.* the ones that would be obtained if $n(t)$ were used.

We now have a new technique that can provide a value for n that will at worst provide the same level of accuracy as the HBIM or RIM methods alone and when $n = n(t)$ will lead to more accurate results. Furthermore, it is easier to apply. With this in mind we now move onto the second main drawback of the integral methods, namely the approximation for time dependent boundary conditions.

3. Time dependent boundary conditions

To understand the problem with time-dependent boundary conditions, consider the two apparently similar cases, $h(t) = t$ and $h(t) = 1 - t$. The exact solutions for each case are plotted in Fig. 2. For $h = t$ (in Fig. 2(a)) the temperature at $x = 0$ increases steadily. For $x > 0$ it decreases monotonically to the far-field temperature $T = 0$. This form of temperature profile is easily dealt with by the integral methods. When $h = 1 - t$ (in Fig. 2(b)) the temperature at $x = 0$ decreases with time. For $x > 0$, the temperature initially decreases monotonically to the far field temperature. However, as the initial energy propagates into the material there comes a time when the internal temperature is greater than that at the boundary. This first occurs after $T_x(0,t) = 0$ and subsequently there is a maximum at $x = p(t)$ where $0 < p < \delta$ (as seen on the curves when $t \geqslant 0.8$). For the problem with $T(0,t) = h(t)$ the standard polynomial profile has the form

$$T = h(t)\left(1 - \frac{x}{\delta}\right)^n, \tag{27}$$

which only permits $T_x = 0$ at $x = \delta$. Consequently it is not appropriate for any temperature profile with a turning point at $x \neq \delta$ and therefore cannot represent the results shown in Fig. 2(b).

3.1. An improved approximating function

To permit a turning point for $x < \delta$, in an analysis of travelling wave solutions to the Korteweg–de Vries equation, Myers and Mitchell [22] used a profile of the form

$$T = a_m(t)\left(\frac{\delta - x}{\delta - p}\right)^m + a_n(t)\left(\frac{\delta - x}{\delta - p}\right)^n, \tag{28}$$

with $m = 3$, $n = 4$. For the current problem we note that, without specifying m, n, this profile satisfies $T = T_x = 0$ at $x = \delta$ provided m, $n > 1$ and also permits $T_x = 0$ for $x \neq \delta$. The condition $T_x(p,t) = 0$ leads to

$$a_m(t) = -\frac{n}{m}a_n(t). \tag{29}$$

We also require $p = 0$ when $T_x(0,t) = 0$, and the above condition is consistent with this requirement. Note that since the temperature profile is defined over a growing region, $p(0) = \delta(0) = 0$. The form of $p(t)$ is discussed in more detail later.

Imposing the boundary condition (3) determines

$$a_n(t) = \frac{h(t)}{y^n - \frac{n}{m}y^m}, \quad \text{where } y = \frac{\delta}{\delta - p}, \tag{30}$$

which means that the temperature, Eq. (28), may be written as

$$T = \frac{h(t)}{y^n - \frac{n}{m}y^m}\left[y^n\left(1 - \frac{x}{\delta}\right)^n - \frac{n}{m}y^m\left(1 - \frac{x}{\delta}\right)^m\right]. \tag{31}$$

We are now left with a problem involving four unknowns, namely the exponents m, n, $\delta(t)$ and $y(t)$ (or $p(t)$). In applying the method of [20] to the current problem, the error E_{mn} varies with both m and n, but otherwise the calculation is the same. Plotting the surface $E_{mn}(m,n)$ shows that the minimum occurs when $m \approx n$. However, setting $m = n$ in the profile defined by (31) leads to $T = 0$ (except at the point $x = 0$). Consequently we write $m = n + \epsilon$, for some $\epsilon \ll 1$, and so the expression (31) reduces to

$$T = \frac{h(t)}{1 - \frac{n}{n+\epsilon}y^\epsilon}\left[1 - \frac{ny^\epsilon}{n+\epsilon}\left(1 - \frac{x}{\delta}\right)^\epsilon\right]\left(1 - \frac{x}{\delta}\right)^n, \tag{32}$$

which can be expanded for small ϵ to give

$$T = h(t)\left(1 - \frac{x}{\delta}\right)^n\left[1 + \frac{\ln\left(1 - \frac{x}{\delta}\right)}{\ln y - \frac{1}{n}} + \mathcal{O}(\epsilon)\right]. \tag{33}$$

Motivated by this expansion we therefore consider a profile of the form

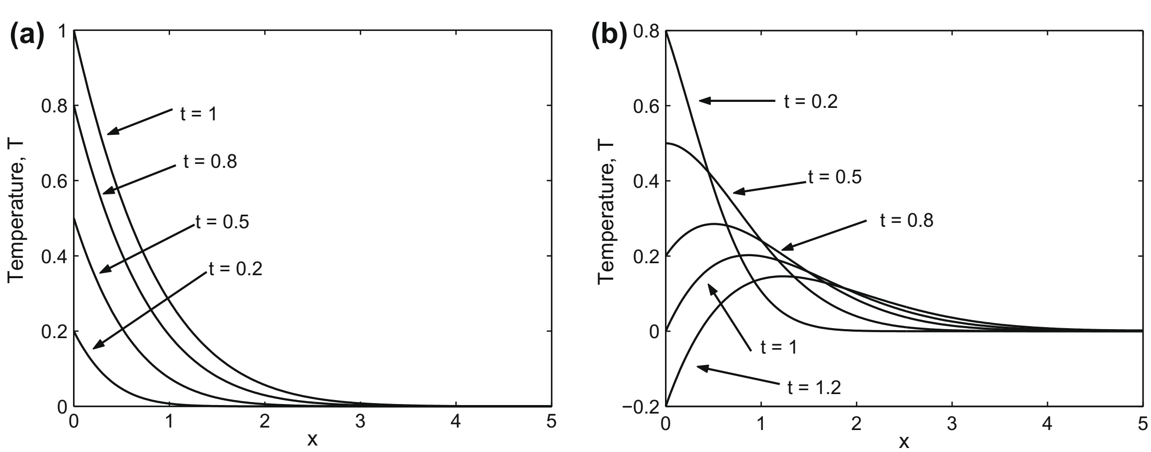

Fig. 2. The exact solution of (1)–(4) at various times for (a) $h(t) = t$ and (b) $h(t) = 1 - t$.

$$T = h(t)\left(1-\frac{x}{\delta}\right)^n\left[1+\phi(t)\ln\left(1-\frac{x}{\delta}\right)\right]. \quad (34)$$

The profile (34) satisfies the boundary condition (3), and the requirements $T_x = 0$ at both $x = \delta$ and $x = p$. Also, the condition $T_{xx}(\delta, t) = 0$ is automatically satisfied provided $n > 2$ (this can be shown by applying L'Hôpital's rule to the product $(1 - x/\delta)^{n-2}$ ln $(1 - x/\delta)$).

Following a similar argument to that in (8) we use the boundary condition $T(0,t) = h(t)$ to deduce that

$$\frac{dh}{dt} = \frac{\partial T}{\partial t}(0,t) = \frac{\partial^2 T}{\partial x^2}(0,t). \quad (35)$$

Substituting (34) into (35) then leads to the expression

$$n(n-1)h + (2n-1)h\phi = h_t\delta^2,$$

and so

$$\phi = \frac{h_t\delta^2 - n(n-1)h}{2n-1}. \quad (36)$$

Note that the standard form for the temperature, Eq. (27), is retrieved by setting $\phi = 0$ in (34). The advantage of this formulation over that of (34) is that when calculating the numerical solution of the governing differential equations the definition of y at $t = 0$ is $y = 0/0$: this obviously leads to numerical difficulties. Whereas $\phi(0) = -n(n-1)h(0)/(2n-1)$ behaves more sensibly. As with the standard profile, (27), this profile now only involves two unknowns n and δ. In deriving (34) we have eliminated the other two unknowns m and $y(t)$ (or $p(t)$): the former by expanding the profile (32) for $m = n + \epsilon$ with $\epsilon \ll 1$ and only keeping the leading order term, and the latter by imposing the extra condition (35).

For a general $h(t)$, and without assuming n constant, substituting for the temperature profile (34) in the HBIM and RIM Eqs. (11) and (13) leads to

$$\text{HBIM:}\quad \frac{d}{dt}\left[\frac{h\delta}{n+1} - \frac{\delta\phi}{(n+1)^2}\right] = \frac{nh+\phi}{\delta}, \quad (37)$$

$$\text{RIM:}\quad \frac{d}{dt}\left[\frac{h\delta^2}{(n+1)(n+2)} - \frac{(2n+3)\delta^2\phi}{(n+1)^2(n+2)^2}\right] = h. \quad (38)$$

These are the two equations that now define the problem for a time-dependent boundary condition. With $\phi = 0$, $h = 1$ we retrieve Eqs. (21) and (22). The analysis can now proceed as before: for the new method we solve these two equations simultaneously, whereas for the minimisation method we solve either (37) or (38) coupled to the error integral (10).

Let us now return to the problem considered in Section 2.1 with $h(t) = 1$. Although this is a case where the standard HBIM profile (27) does give a good approximation, we will show that even for this simple test case, the profile (34) leads to a much more accurate solution. If $h(t) = 1$ then $\phi = -n(n-1)/(2n-1)$ (defined in (36)) and the above equations reduce considerably. The introduction of ϕ into the problem, in this case, merely changes the coefficient of δ or δ^2 in the derivative terms of (37) and (38) from those of (21) and (22). Hence the equations again allow constant n solutions and $\delta = \alpha\sqrt{t}$ where

$$\text{HBIM:}\quad \alpha = \sqrt{\frac{2n^2(n+1)^2}{3n^2-1}},$$

$$\text{RIM:}\quad \alpha = \sqrt{\frac{(2n-1)(n+1)^2(n+2)^2}{2(2n+1)(n^2+n-1)}}. \quad (39)$$

Equating these two expressions gives one positive root, $n = 5.5132$. Using the minimisation technique (noting that for T defined by (34) this is a non-trivial exercise) predicts $n \approx 5.2895$, 5.5215 for the HBIM and RIM, respectively.

In Fig. 3 we compare temperature profiles for $h(t) = 1$ predicted by the exact solution (6) and the polynomial approximation (7) where n is determined by the new method and the minimisation technique. The right hand plot shows the absolute errors, which never exceed 0.04 (corresponding to a 4% error). The RIM with a minimised n is clearly the most accurate, the HBIM the least. As mentioned earlier, the new method generally gives errors bounded by the two other methods. In this case it appears that the new method benefits from the accuracy of the RIM to outperform the HBIM solution. Note that, for this case only, the new method corresponds to the classical HBIM solution where $n = 2$. In Fig. 4 we show the corresponding results for the logarithmic temperature profile (34). Obviously it is difficult to distinguish between the temperature profiles shown in Fig. 4(a), consequently in Fig. 4(b) we show the absolute errors. These errors are significantly lower than for the polynomial profile and we can see that the error never exceeds 0.8%. It is hard to say which is the best method, since for small x the error is least with the minimised HBIM solution but later switches to the minimised RIM. The solution via the new technique closely follows the RIM solution (but is very slightly worse). In all subsequent examples the errors are small, so from now on we will usually only plot the errors rather than the temperature profile. Furthermore, with the exception of the case plotted in Fig. 3(b), n will always be significantly different to 2 and consequently plotting the error for $n = 2$ requires a larger scale for the

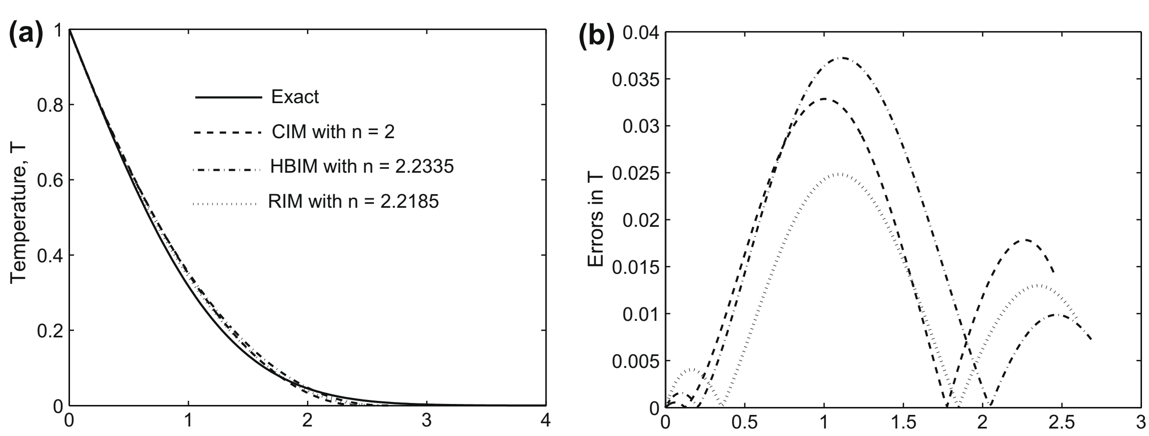

Fig. 3. Results at $t = 0.5$ for $h(t) = 1$ for (a) temperature profiles using (27) for the CIM (dashed), HBIM (dot-dashed) and RIM (dotted), with the latter two using n which minimises E_n and (b) corresponding absolute errors.

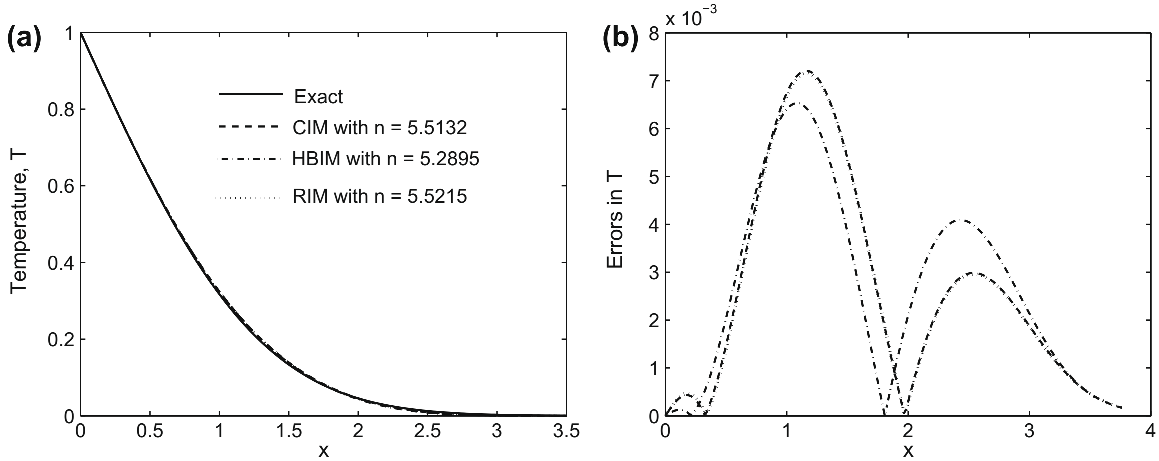

Fig. 4. Results at $t = 0.5$ for $h(t) = 1$ for (a) temperature profiles using (34) for the CIM (dashed), HBIM (dot-dashed) and RIM (dotted), with the latter two using n which minimises E_n and (b) corresponding absolute errors.

figure and makes it difficult to see the difference between the other solutions; hence we will no longer plot the $n = 2$ case.

For the constant flux and cooling boundary conditions the logarithmic profiles are both of the form

$$T = \left[a(t) + b(t)\ln\left(1 - \frac{x}{\delta}\right)\right]\left(1 - \frac{x}{\delta}\right)^n, \tag{40}$$

where for the constant flux case we have

$$a(t) = \frac{(3n^2 - 6n + 2)\delta}{n^2(2n-3)}, \quad b(t) = -\frac{(n-1)(n-2)\delta}{n(2n-3)},$$

and for the cooling condition

$$a(t) = \frac{[3n^2 - 6n + 2 + (2n-1)\delta]\delta}{n^2(2n-3) + 2(2n-1)(n-1)\delta + (2n-1)\delta^2},$$
$$b(t) = -\frac{[n(n-1)(n-2) + n(n-1)\delta]\delta}{n^2(2n-3) + 2(2n-1)(n-1)\delta + (2n-1)\delta^2}.$$

As for the fixed temperature boundary condition, an extra condition is required to determine $a(t)$ and $b(t)$. Instead of the condition (35) we differentiate the boundary conditions $T_x = -1$ and $T_x = T-1$ with respect to t leading to $T_{xxx} = 0$ and $T_{xxx} = T_{xx}$ at $x = 0$ for the constant flux and cooling conditions, respectively.

In Fig. 5 we compare the absolute errors calculated when applying a constant flux boundary condition, $T_x(0,t) = -1$. For the polynomial temperature profile, namely $T = (\delta/n)(1 - x/\delta)^n$, the CIM predicts $n = 4$, and the minimisation technique predicts $n = 3.5848$, 3.8235 for HBIM and RIM, respectively. The worst error is approximately 10^{-2} (or 1% error). Note, for the classical HBIM, with $n = 2$, we find errors of the order 8.5%. For the logarithmic profile, we find $n = 7.5152$ for the CIM, and the minimisation technique predicts $n = 6.9582$, 7.3059 for HBIM and RIM, respectively. The errors are much lower, with the worst approximately 3.5×10^{-3} (or 0.3%). In both figures, the highest error comes from the RIM, while for much of the domain the new method is the most accurate.

In Fig. 6 we compare the absolute errors calculated when applying a cooling boundary condition, $T_x = T - 1$ at $x = 0$. For the polynomial temperature profile, namely $T = (\delta/(n + \delta))(1 - x/\delta)^n$, the minimisation technique predicts $n = 2.8455$, 2.9655 for HBIM and RIM, respectively. The errors are all below 2% (slightly larger than for the constant flux boundary condition). Also, note that the maximum error occurs here for the CIM, rather than the RIM. For the logarithmic profile the minimisation technique predicts $n = 6.1265$, 6.4335 for HBIM and RIM, respectively. The errors are also an order of accuracy better, below 0.4%, and now RIM gives the maximum error.

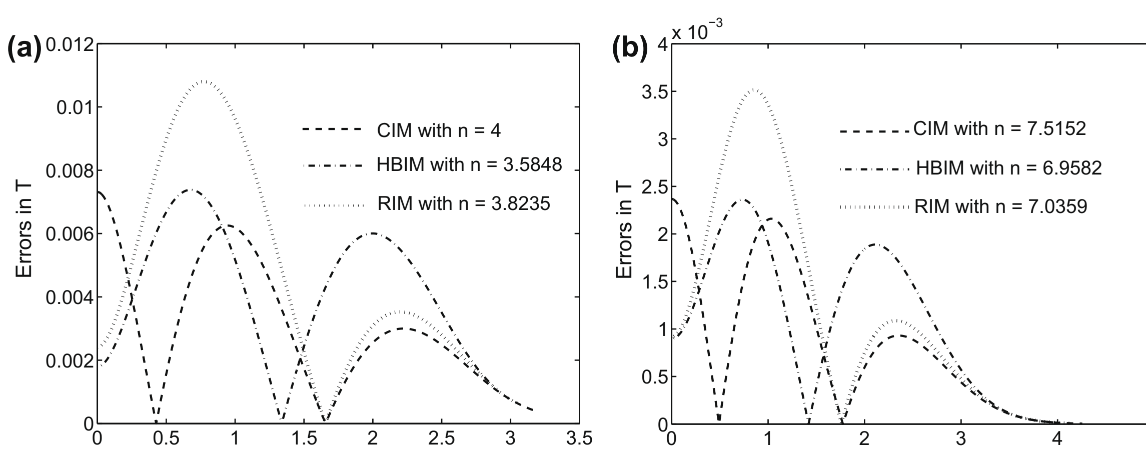

Fig. 5. Comparison of absolute errors at $t = 0.5$ for the constant flux boundary condition, $T_x = -1$, for (a) the polynomial profile for the CIM (dashed), HBIM (dot-dashed) and RIM (dotted) solutions, with the latter two using n which minimises E_n and (b) corresponding absolute errors using the logarithmic profile.

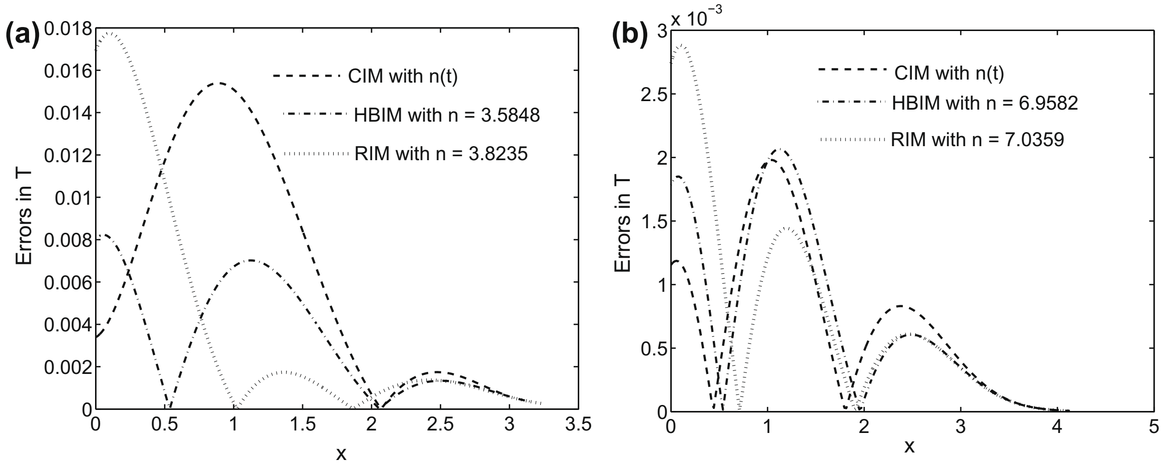

Fig. 6. Comparison of absolute errors at $t = 0.5$ for the cooling condition, $T_x = T - 1$, for (a) the polynomial profile for the CIM (dashed), HBIM (dot-dashed) and RIM (dotted) solutions, with the latter two using n which minimises E_n and (b) corresponding absolute errors using the logarithmic profile.

3.2. Why is the logarithmic profile more accurate?

From the above examples it is clear that the logarithmic approximation improves on the polynomial approximations, typically by an order of magnitude. We can see why the logarithmic profile is more accurate than the polynomial by looking at an expansion of the different solutions. Firstly, we define the standard thermal similarity variable $\xi = x/\sqrt{t}$. Then, an expansion of the exact solution (6) for small ξ gives

$$T \sim 1 - \frac{1}{\sqrt{\pi}}\xi + \frac{1}{12\sqrt{\pi}}\xi^3 - \frac{1}{160\sqrt{\pi}}\xi^5 + \mathcal{O}(\xi^6). \quad (41)$$

Since $\delta = \alpha\sqrt{t}$ for both approximate profiles, we can also expand (27) and (34) in terms of ξ. Thus (27) becomes

$$T = 1 - \frac{n}{\alpha}\xi + \frac{n(n-1)}{2\alpha^2}\xi^2 - \frac{n(n-1)(n-2)}{6\alpha^3}\xi^3 + \mathcal{O}(\xi^4), \quad (42)$$

and (34) is now

$$T = 1 - \frac{n^2}{\alpha(2n-1)}\xi + \frac{n^2(n-1)^2}{6\alpha^3(2n-1)}\xi^3 + \mathcal{O}(\xi^4). \quad (43)$$

The expansions of (43) and the exact solution (41) both skip the $\mathcal{O}(\xi^2)$ term and so should provide close agreement for the correct choice of n. The expansion of the standard HBIM solution (42) includes the $\mathcal{O}(\xi^2)$ term and so is unlikely to behave in a similar manner to the exact solution as ξ increases.

Previous researchers have chosen an approximating function based on the expansion of a known exact solution and consequently the approximation works well for the chosen boundary condition, but it is not very good for other conditions [4,5,16,17,19,23]. With this in mind from the above analysis we can strictly only deduce that the logarithmic profile is more accurate for $h = 1$. This result can be extended to a more general $h(t)$ by expanding $h(t-\tau)$ about t in Eq. (5). An expansion of the resulting expression for $\xi = x/\sqrt{t}$ can be compared with an expansion of the logarithmic approximating function (34), and we find that the $\mathcal{O}(\xi^0, \xi^2)$ terms match exactly. Consequently the logarithmic approximation must provide good agreement with the exact solution, at least for small ξ.

3.3. The boundary condition h = h(t)

Although the logarithmic profile clearly provides more accurate solutions than the standard polynomial for the boundary conditions discussed in the preceding section, the added complexity means that the polynomial will probably be preferred for these problems. However, when h is time dependent and the temperature includes a moving peak then the polynomial form is not appropriate. In this case an alternative form, such as the logarithmic function (34) must be used.

Let us consider the two cases where $h(t) = t$ and $h(t) = 1 - t$. Below we list the appropriate equations for both the standard polynomial temperature profile (27), in which the HBIM and RIM equations are simply (37) and (38) with $\phi = 0$, and the logarithmic temperature profile (34) with $\phi \neq 0$.

- **Boundary condition** $h(t) = t$: Using the integral form in (5) the exact solution is

$$T = \left(t + \frac{x^2}{2}\right)\operatorname{erfc}\frac{x}{2\sqrt{t}} - \sqrt{\frac{t}{\pi}}xe^{-x^2/4t}. \quad (44)$$

For profile (27), we note that (38), with $\phi = 0$, integrates immediately to determine δ. Using this solution in (37) we find that solutions with a constant n are supported, so we can also integrate this equation to obtain

$$\text{HBIM}: \quad \delta = \sqrt{\frac{2n(n+1)t}{3}}, \quad \text{RIM}: \quad \delta = \sqrt{\frac{(n+1)(n+2)t}{2}}. \quad (45)$$

Equating these two expressions leads to $n = 6$, whereas finding n to minimise E_n gives $n = 5.4015$, 5.7595 for the HBIM and RIM, respectively.

For the logarithmic profile (34) (and assuming that n is constant), solving the HBIM and RIM Eqs. (37) and (38), gives $\delta = \alpha\sqrt{t}$ where α satisfies the following equations quadratic in α^2:

$$\text{HBIM}: \ 3\alpha^4 - (7n^2 - 4n - 5)\alpha^2 + 2n^2(n+1)^2 = 0, \quad (46)$$

$$\text{RIM}: \ 2(2n+3)\alpha^4 - 4(2n+1)(n^2+n-1)\alpha^2 + (2n-1)(n+1)^2(n+2)^2 = 0. \quad (47)$$

Solving these equations and equating the expressions for α^2 leads to $n = 10.7204$, whereas finding n to minimise E_n gives $n = 10.1413$, 10.1810 for the HBIM and RIM, respectively.

Results for the two approximating functions are given in Fig. 7(a) and (b). The polynomial profile clearly shows excellent agreement with the exact solution, with an error typically below 0.3%. However, the logarithmic approximation, Eq. (34), shows errors an order of magnitude less, typically below 0.03%. Of course

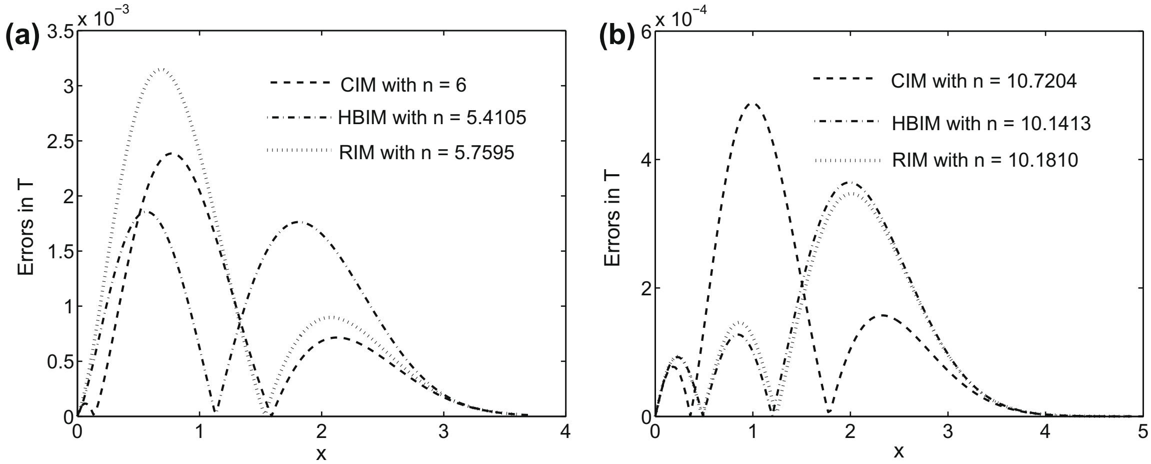

Fig. 7. Comparison of absolute errors at $t = 0.5$ for boundary condition $h(t) = t$ for the CIM (dashed), HBIM (dot-dashed) and RIM (dotted) solutions with (a) the polynomial profile (27) and (b) the logarithmic profile (34).

both sets of errors are so low that the polynomial profile could be used with confidence.

- **Boundary condition** $h(t) = 1 - t$: Using the integral form in (5) the exact solution is given by

$$T = \left(1 - t - \frac{x^2}{2}\right)\mathrm{erfc}\frac{x}{2\sqrt{t}} + \sqrt{\frac{t}{\pi}}xe^{-x^2/4t}. \tag{48}$$

For profile (27), with $\phi = 0$, the RIM Eq. (38) integrates immediately. The HBIM, Eq. (37), may be integrated under the assumption of constant n, leading to

$$\text{HBIM: } \delta = \sqrt{\frac{2n(n+1)}{3(1-t)^2}[1-(1-t)^3]}, \tag{49}$$

$$\text{RIM: RIM: } \delta = \sqrt{\frac{(n+1)(n+2)}{2(1-t)}[1-(1-t)^2]}. \tag{50}$$

Equating these two expressions shows that $n = n(\delta(t)) = n(t)$, so the assumption used to derive the HBIM result is incorrect and this problem requires more careful consideration. Consequently we now move on to dealing with a time-dependent n.

3.4. Time dependent power, $n(t)$

Now consider applying the new method for a general time dependent boundary condition; we assume $n = n(t)$ and therefore solve both (37) and (38) simultaneously. In the case of profile (27), *i.e.* with $\phi = 0$, these become

$$h\frac{d\delta}{dt} - \frac{h\delta}{n+1}\frac{dn}{dt} = \frac{n(n+1)h}{\delta} - h_t\delta, \tag{51}$$

$$2h\frac{d\delta}{dt} - \frac{(2n+3)h\delta}{(n+1)(n+2)}\frac{dn}{dt} = \frac{(n+1)(n+2)h}{\delta} - h_t\delta, \tag{52}$$

which may be expressed as

$$\delta\frac{d\delta}{dt} = (n+1)\left[(4+n-n^2) + \frac{\delta^2 h_t}{h}\right], \tag{53}$$

$$\delta^2\frac{dn}{dt} = (n+1)(n+2)\left[(n+1)(2-n) + \frac{\delta^2 h_t}{h}\right]. \tag{54}$$

When $h = 1$ these reduce to (23), with n constant. For general $h(t)$ and time dependent $n(t)$ we must now determine the initial condition $n(0)$.

For $t \to 0$ we assume $h \to At^\beta$ (since our focus is on physically realistic forms of h this assumption seems reasonable). We also assume that the value of n tends to a non-zero constant $n \to n_0$. Substituting for h into Eq. (38), with $\phi = 0$, and integrating gives

$$\frac{\delta^2}{(n_0+1)(n_0+2)} = \frac{t}{\beta+1}, \tag{55}$$

and so $\delta = \sqrt{\alpha t}$ where $\alpha = (n_0+1)(n_0+2)/(\beta+1)$. Substituting for h, δ and n into Eq. (37) gives

$$n_0(n_0+1) = \alpha\left(\beta + \frac{1}{2}\right). \tag{56}$$

Note that this equation is independent of t indicating the assumption that $n \to n_0$ is valid. With the definition of α given above we find $n_0 = 2 + 4\beta$. For boundary conditions such as $h(t) = 1, 1-t, \cos t, e^t$, the limit $t \to 0$ gives $h \to 1$ and so $\beta = 0$ and $n_0 = 2$. For conditions such as $h = t, \sin t$ *etc.*, $h \to t$ as $t \to 0$ and $n_0 = 6$.

When ϕ is non-zero we again assume $h \to At^\beta$, $n \to n_0$. To prevent ϕ from blowing up, where

$$\phi = At^\beta\left[\frac{\beta\delta^2/t - n_0(n_0+1)}{2n_0-1}\right], \tag{57}$$

requires $\delta = \sqrt{\alpha t}$ and so

$$\phi = \gamma h, \quad \gamma = \frac{\beta\alpha - n_0(n_0+1)}{2n_0-1}. \tag{58}$$

Substituting into Eqs. (37) and (38) then gives two simultaneous equations for n_0 and α

$$\alpha(2\beta+1)[n_0+1-\gamma] = 2(n_0+\gamma)(n_0+1)^2, \tag{59}$$

$$\alpha(\beta+1)[(n_0+1)(n_0+2) - \gamma(2n_0+3)] = (n_0+1)^2(n_0+2)^2. \tag{60}$$

For $\beta = 0$ we find $n_0 = 5.513$, $\alpha = 28.594$, for $\beta = 1$, $n_0 = 10.720$, $\alpha = 52.775$, for $\beta = 2$, $n = 15.73$, $\alpha = 75.68$, for $\beta = 3$, $n_0 = 20.618$, $\alpha = 97.824$.

Returning to the boundary condition $h = 1 - t$ discussed in the previous section, with $\phi = 0$ the problem is governed by Eqs. (37) and (38),

$$\frac{d}{dt}\left[\frac{(1-t)\delta}{n+1}\right] = \frac{n(1-t)}{\delta}, \quad \frac{d}{dt}\left[\frac{(1-t)\delta^2}{(n+1)(n+2)}\right] = (1-t). \tag{61}$$

Obviously the second equation may be integrated analytically, but the first must be evaluated numerically. Since $h \to 1$ as $t \to 0$ the initial conditions are $n(0) = 2$, $\delta(0) = 0$. With $\phi \neq 0$ we solve the full versions of (37) and (38) with initial conditions $n(0) = 5.5132$ (see (39)).

As an example of a nonlinear boundary condition, where the solution again breaks down using the standard integral methods, we now consider $h(t) = t(1-t)$. The solution of (5) gives

$$T = \left(t - t^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4 - tx^2\right)\mathrm{erfc}\frac{x}{2\sqrt{t}} + \sqrt{\frac{t}{\pi}}x\left(\frac{1}{6}x^2 + \frac{10}{3}t - 1\right)e^{-x^2/4t}. \quad (62)$$

The appropriate differential equations for δ and n are (37) and (38) with $h = t(1-t)$, subject to $n(0) = 6$, $\delta(0) = 0$.

Fig. 8 displays the errors in the temperature obtained using the polynomial and logarithmic approximating functions when $h = 1 - t$ at time $t = 0.5$. Note that the error for the CIM, when using the polynomial approximation, is unacceptably high at around 9%. This is our first indication that the method is breaking down due to the time dependence of the boundary condition. The logarithmic profile gives a much more acceptable maximum error around 1%.

The problem can be seen clearly in Fig. 9(a) which displays plots of the temperature with the boundary condition $h = 1 - t$ at times $t = 0.2, 0.6$. For $t = 0.2$ the agreement is reasonable, although the errors are significantly larger than for previous examples. At $t = 0.6$ a maximum has appeared in the exact solution and all approximate solutions give very poor agreement. The CIM is by far the worst. The reason for this is that in this method n is a decreasing function of time with $n(0.2) \approx 1.37$, $n(0.6) \approx 0.019$. Recall that to ensure $T_x(\delta, t) = 0$ requires $n > 1$, and this condition is clearly violated at some time before $t = 0.6$. In fact, the reason for showing the $t = 0.6$ solution is to highlight how the method breaks down: very shortly afterwards this time n becomes negative and $T(\delta, t) \to \infty$. The other two approximate methods do not break down in the same way since n is fixed. However, none of the solutions can capture the peak. Furthermore, at $t = 1$, when $h(t) = 0$ all three approximate methods predict $T = 0\ \forall\ x > 0$. The classical approximating profile, with $n = 2$, will behave in a similar manner to the HBIM and RIM curves shown as dash-dotted and dotted lines.

Fig. 9(b) displays temperatures predicted by the polynomial approximation for the boundary condition $h = t(1-t)$ at times $t = 0.2, 0.825$. In this case the solution for the CIM blows up soon after $t = 0.825$. Again it is clear that at $t = 0.2$ the agreement is reasonable but for larger times the similarity decreases and in the curves for $t = 0.825$ show terrible agreements.

The temperature profiles for the same boundary conditions as shown in Fig. 9, but with the logarithmic approximating function, are given in Fig. 10. In this case we show results for the single time $t = 1$, which is where the polynomial approximating functions predict a zero temperature. Both figures show that the agreement is very good and the peak is accurately captured. This correspondence continues for larger times. In general the results displayed throughout this paper indicate that the logarithmic profile is more accurate than a polynomial. The current result also proves that it is more versatile and can deal with time dependent boundary conditions which cause the standard method to break down rapidly.

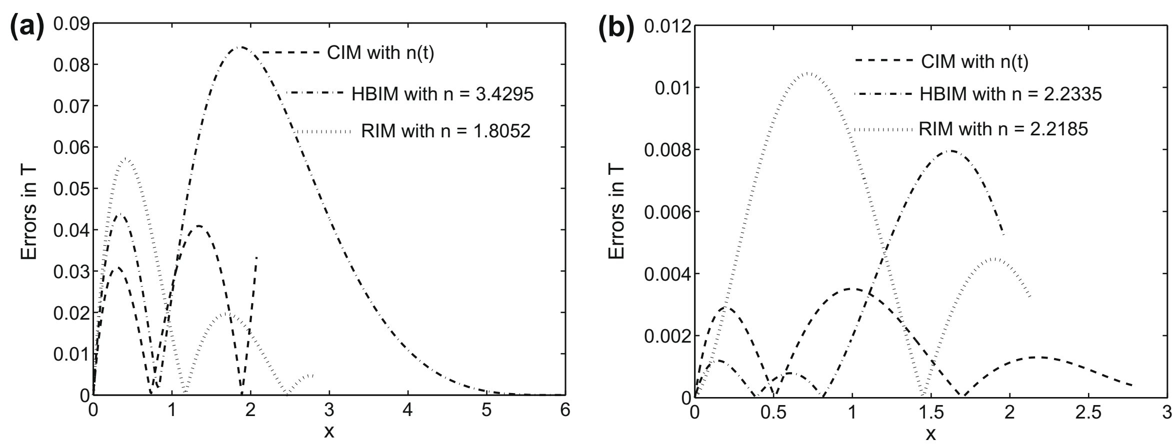

Fig. 8. Comparison of absolute errors at $t = 0.5$ for boundary condition $h(t) = 1 - t$ for the CIM (dashed), HBIM (dot-dashed) and RIM (dotted) solutions with (a) the polynomial profile (27) and (b) the logarithmic profile (34).

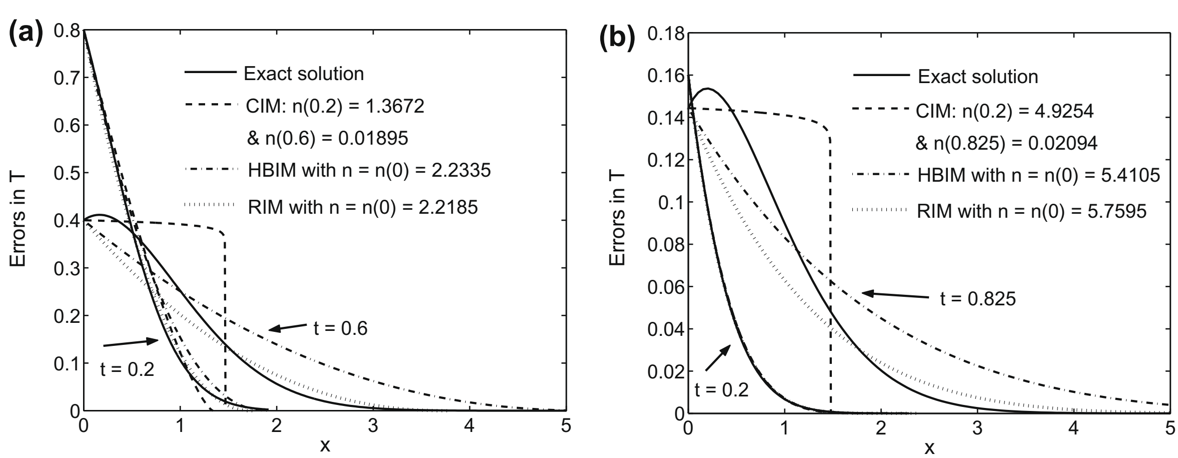

Fig. 9. Comparison of exact (solid line), CIM (dashed), HBIM (dash-dot) and RIM (dotted) with a polynomial approximating function for (a) $h = 1 - t$ and (b) $h = t(1 - t)$.

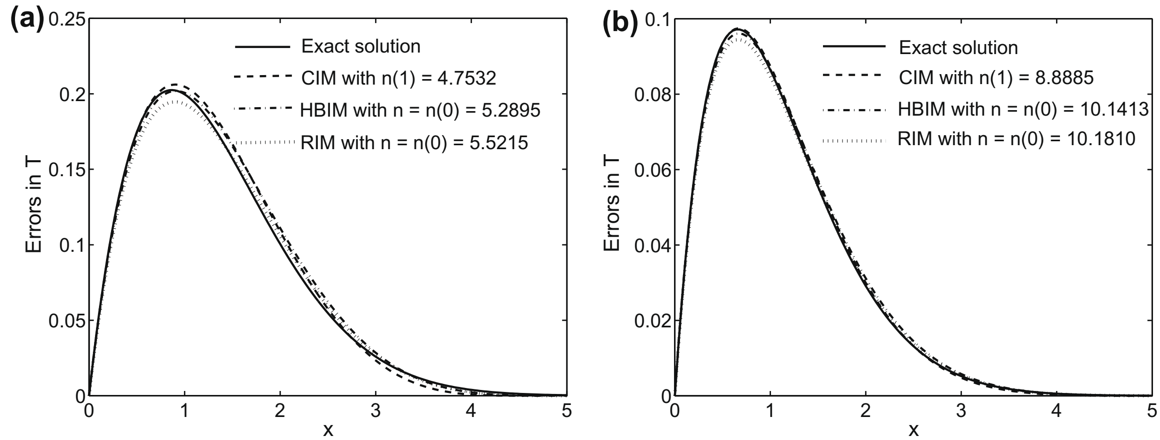

Fig. 10. Comparison of exact (solid line), CIM (dashed), HBIM (dash-dot) and RIM (dotted) with a logarithmic approximating function for (a) $h = 1 - t$ and (b) $h = t(1 - t)$.

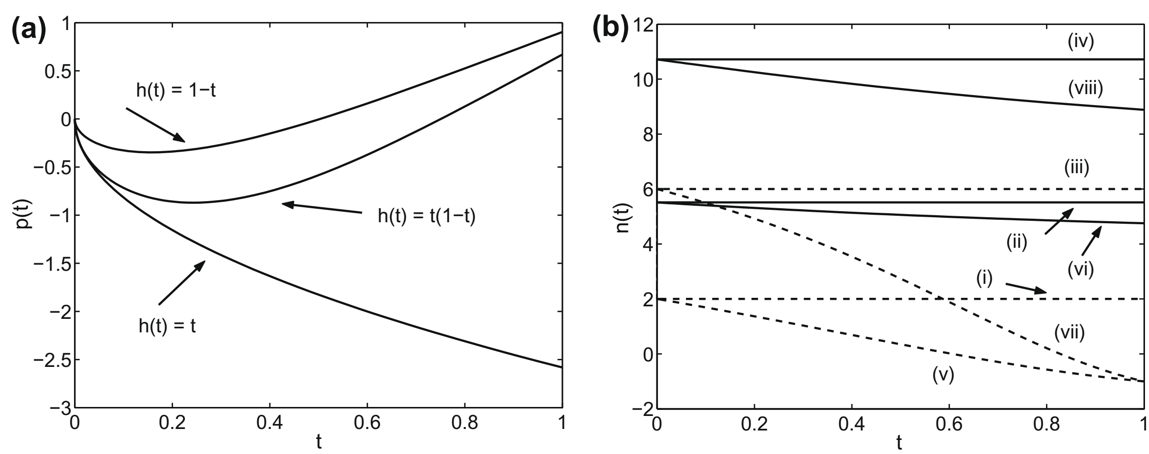

Fig. 11. Time dependent boundary conditions: (a) plots of the peak $p(t)$ against t for profile (34) and (b) $n(t)$.

Finally, in Fig. 11 the variation of the position of the peak $p(t)$ and the exponent $n(t)$ are displayed. The peak position, shown in Fig. 11(a), is determined from (30) to give $p = \delta(1 - 1/y)$ where y can be found using the expression $\phi(\ln y - 1/n) = 1$ (which comes from comparing (33) and (34)). Initially $p = 0$. For the boundary condition $h = \mathrm{t}$ the peak moves backwards and $p(t) < 0$ for all time. Hence the polynomial approximating functions can model the temperature, although the logarithmic profile will be more accurate. When $h = 1 - t$ and $h = t(1 - t)$, initially the peak moves backwards but in both cases $p(t)$ becomes positive for finite time and the polynomial approximating functions fail soon afterwards. Fig. 11(b) shows the variation of n for the polynomial (dashed lines) and logarithmic (solid lines) approximations for (i) and (ii) $h = 1$, (iii) and (iv) $h = \mathrm{t}$, (v) and (vi) $h = 1 - \mathrm{t}$, (vii) and (viii) $h = t(1 - t)$. The lines (i)–(iv) all show that n is constant. Curves (v) and (vii) show n decreasing with t and in both cases it becomes negative for finite time. As soon as n becomes negative the polynomial solutions will break down (although they will be inaccurate for some time prior to this). Consequently, the polynomial approximation may only be used for small times for the boundary conditions $h = 1 - t$, $t(1 - t)$. The corresponding curves for the logarithmic profile, (vi) and (viii), both have $n > 0$ (and in fact $n > 4$ so ensuring a zero gradient at $x = \delta$) and so do not break down. Our calculations for larger times show that n remains above 2 in both cases and tends to a positive asymptote.

4. Conclusions

The method described in this paper improves on standard heat balance integral methods in two significant ways. Firstly, a new approach has been presented to calculate the value of the exponent n in the temperature approximation. A method of this type has recently been developed, [20], where n is determined by minimising an error associated with either the HBIM or RIM solutions. The method presented in this paper combines the HBIM and RIM solutions to determine n in a different manner. The results show that solutions have a similar level of accuracy to the minimisation technique and in some cases are an improvement. The new method also has the advantage that the algebra is considerably simpler. Furthermore, when $n = n(t)$, the complexity of the analysis for the minimisation technique means that a constant value of n has to be taken and so the accuracy deteriorates. With the present method there is no such problem, n simply satisfies a first order differential equation and results show a high degree of accuracy. The new method is therefore particularly useful for time dependent boundary conditions.

The second step forward is in the analysis of problems where the temperature has a moving peak. Previously, standard heat balance methods could not model this situation. Solutions in the literature were only provided for small times, before the method broke down. The logarithmic approximating function presented here can predict and follow a single moving peak. Even for boundary conditions where the standard polynomial approximation can provide solutions for all time, the logarithmic approximation proved to be more accurate. The only downside to this new approximation is that the analysis is more complicated, and the main appeal of the heat balance methods is in their simplicity. However, given that standard heat balance methods cannot deal with time dependent boundary conditions it seems worth investing some time in formulating the new method.

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