



UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4006

SEMESTER: Spring 2005

MODULE TITLE: Engineering Mathematics 5

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. Helen Purtill

PERCENTAGE OF TOTAL MARKS: 100%

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Answer any 4 questions.**

1. (a) Consider the vector valued function

$$\mathbf{r}(t) = \begin{cases} (t, 1+t, 0) & \text{if } -1 \leq t \leq 0 \\ (\sin t, e^t, t^2) & \text{if } 0 \leq t \leq 1 \end{cases}$$

find its derivative, and hence find the unit tangent vector to the curve defined by $\mathbf{r}(t)$ and comment on its smoothness

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- (b) Sketch the curve $\mathbf{r}(t) = (t+1, t)$ where $-2 \leq t \leq 0$ in two dimensions and find an expression for its arclength from $t = -2$ to an arbitrary point on the curve. Hence find the arclength from $t = -2$ to $t = -1$. Write the equation for the curve in intrinsic form.

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- (c) A particle P moves on a disk towards the edge with position vector

$$\mathbf{r}(t) = (t^2 + t)\mathbf{b}$$

where \mathbf{b} is a unit vector rotating together with the disk with constant angular speed ω in the counterclockwise sense. Find a form for the unit vector \mathbf{b} and the acceleration of the particle P .

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2. (a) Find the work done in moving a particle in a force field given by

$$\mathbf{F}(x, y, z) = 4x^2y\mathbf{i} - z\mathbf{j} + y^3xz\mathbf{k},$$

along the curve with parametric definition $\mathbf{r} = (t, t^2, 1)$ where $0 \leq t \leq 1$.

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- (b) Calculate the volume of a hemisphere using spherical coordinates.
- (c) State Green's Theorem in the plane. Using Green's theorem evaluate the line integral $\oint \mathbf{F} \cdot d\mathbf{r}$ counterclockwise around the boundary of C of the region R , where $\mathbf{F} = x^2y\mathbf{i} - y^2x\mathbf{j}$ and $R : x^2 + y^2 \leq 25$, $y \geq 0$.

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3. (a) Find the directional derivative of the scalar valued function

$$f(x, y, z) = 3x^2y + z^3x - 4x$$

at the point $(1, -1, 0)$ in the direction of $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

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- (b) Using the fact that $\nabla\Omega$ (where $\Omega = \Omega(x, y)$ is a scalar function of position) is perpendicular to its own level curves, find a unit normal to the curve defined by $y - x^2 + 1 = 0$ at the point $(0, -1)$. Verify this result with a rough sketch.

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- (c) Show that the vector field $\mathbf{v} = -g\mathbf{k}$ (where g is the (constant) acceleration due to gravity) is a conservative vector field. Determine an associated *scalar potential* ϕ for this vector field \mathbf{v} . 5
- (d) Use Taylor's theorem in two dimensions to find a first order approximation for $f(1,0.5)$ based on quantities evaluated at the point $(1.5,0.8)$ in the case where

$$f(x, y) = x^2y + y.$$

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- 4. (a) Explain the term *linearity principle* for a linear homogeneous partial differential equation. 4
- (b) Consider the partial differential equation

$$A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} = f(x, y, u_x, u_y)$$

where A, B, C and f are all known functions. Explain the idea of *characteristics* of a partial differential equation. Noting that the equation defined above has characteristics defined by

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

suggest a way to classify equations as parabolic, elliptic or hyperbolic. Fully classify the following equations. Find and sketch the real characteristics (if any) of the following partial differential equations

$$\begin{aligned} u_{xx} + u_{xy} - 2u_{yy} &= 0 \\ \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + x^2 &= 0 \\ xu_{xy} + 2u &= 0 \end{aligned}$$

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- (c) Solve the following pseudo differential equations for $u = u(x, y)$:

$$u_x - 2 = 0$$

$$u_x + u^2 = 0$$

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- (d) Verify that $u(x, y) = \cos 3t \sin x$ is a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

for a suitable value of the constant c . Identify this value of c .

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5. (a) A cave (located at $x = l$) containing a quantity of ice is located at a distance l below the surface of the earth ($x = 0$). Assume that the initial temperature distribution in the region beneath the surface of the earth is $u(x, 0) = g(x)$, $0 \leq x \leq l$. Assume also that the temperature of surface of the earth and of the cave is maintained at 0°C .

Assume that the flow of heat in the region $0 \leq x \leq l$ can be modelled by linear heat equation $u_t = c^2 u_{xx}$. Set up the appropriate initial boundary value problem. Use the method of separation of variables to solve this problem.

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- (b) Consider the equation

$$4u_{xx} - u_{yy} = 0$$

Using the transformation $\alpha = x + 2y$, $\beta = x - 2y$ reduce the equation to a simpler form and determine the particular solution that satisfies the boundary conditions

$$u(x, 0) = \sin x \quad u_y(x, 0) = 0$$

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6. Let Ω be the unit square $(0, 1) \times (0, 1)$, with boundary Γ and closure $\bar{\Omega} = [0, 1] \times [0, 1]$. Consider the boundary value problem

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega$$

with boundary conditions

$$u(x, y) = g(x, y), \quad (x, y) \in \Gamma$$

- (a) Taking a fixed-width mesh h in both the x and y directions, formulate the discretised problem and outline a uniform mesh for this problem. The 2nd derivative can be approximated using the standard $O(h^2)$ operator.

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- (b) Show that the finite difference operator L^h satisfies a maximum principle of the form:

$$L^h v_{i,j} \geq 0 \quad \forall i, j \Rightarrow \max_{\Omega^h} |v_{i,j}| \leq \max_{\Gamma^h} |v_{i,j}|$$

where $v_{i,j}$ is any mesh-function defined on $\bar{\Omega}^h$.

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- (c) Using the maximum principle, show that the solution of the discretised problem is unique.
- (d) Explain how an estimate of the upper bound for any mesh function $v_{i,j}$ of the form

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$$\max_{\Omega^h} |v_{i,j}| \leq \max_{\Gamma^h} |v_{i,j}| + \frac{1}{2} \max_{\Omega^h} |L^h v_{i,j}|$$

can be employed to comment on the stability and convergence of the numerical method.

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