MA4006: Spring 2012 Exam solutions

Sarah Mitchell

SECTION A

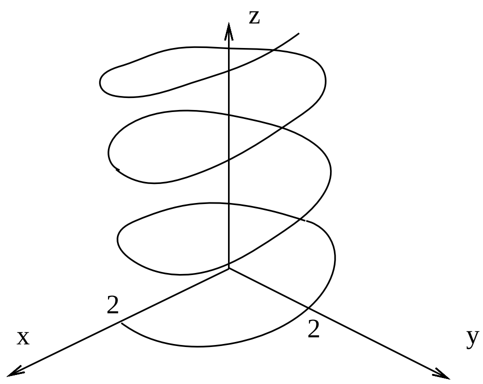
Question 1

(a) Since $\hat{\mathbf{u}}$ is a unit vector it follows that

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = 1 \quad \Rightarrow \quad \frac{\mathrm{d}\hat{\mathbf{u}}}{\mathrm{d}t} \cdot \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \frac{\mathrm{d}\hat{\mathbf{u}}}{\mathrm{d}t} = 0 \quad \Rightarrow \quad 2\hat{\mathbf{u}} \cdot \frac{\mathrm{d}\hat{\mathbf{u}}}{\mathrm{d}t} = 0,$$

which implies that $\hat{\mathbf{u}}$ is perpendicular to $\frac{d\hat{\mathbf{u}}}{dt}$.

(b) (i)



(b) (ii) Since $\mathbf{r} = (2\cos 2t, 2\sin 2t, 3t)$, we have

$$\mathbf{v} = \mathbf{r}' = (-4\sin 2t, 4\cos 2t, 3), \qquad \mathbf{a} = \mathbf{v}' = (-8\cos 2t, -8\sin 2t, 0).$$

To show they are perpendicular we must show that $\mathbf{v} \cdot \mathbf{a} = 0$. So

$$\mathbf{v} \cdot \mathbf{a} = (-4\sin 2t, 4\cos 2t, 3) \cdot (-8\cos 2t, -8\sin 2t, 0) = 32\sin 2t\cos 2t - 32\sin 2t\cos 2t = 0.$$

(b) (iii) The arclength is

$$s = \int_0^t |\mathbf{r}'(t_0)|\,\mathrm{d}t_0 = \int_0^t |\mathbf{v}(t_0)|\,\mathrm{d}t_0.$$

Now, $|\mathbf{v}(t)| = \sqrt{4^2 + 3^2} = 5$ and so

$$s = \int_0^t 5\,\mathrm{d}t_0 = 5\,t.$$

(b) (iv) From part (iii) we have $t = s/5$. Then the intrinsic equation can be written as

$$\mathbf{r}(s) = \left(2\cos\frac{2s}{5}, 2\sin\frac{2s}{5}, \frac{3s}{5}\right).$$

The curvature $\kappa(s)$ is given by

$$\kappa(s) = \left|\mathbf{r}''(s)\right|,$$

and so

$$\begin{aligned}
\mathbf{r}'(s) &= \left(-\frac{4}{5}\sin\frac{2s}{5}, \frac{4}{5}\cos\frac{2s}{5}, \frac{3}{5}\right) \\
\mathbf{r}''(s) &= \left(-\frac{8}{25}\cos\frac{2s}{5}, -\frac{8}{25}\sin\frac{2s}{5}, 0\right) \\
\Rightarrow \quad \kappa(s) &= \sqrt{\left(-\frac{8}{25}\cos\frac{2s}{5}\right)^2 + \left(-\frac{8}{25}\sin\frac{2s}{5}\right)^2} = \frac{8}{25}.
\end{aligned}$$

Also, the radius of curvature is $\rho = 1/\kappa(s) = 25/8$.

(c) The unit vector in the direction $(1, 1, -2)$ is

$$\frac{(1,1,-2)}{\sqrt{1^2 + 1^2 + (-2)^2}} = \frac{1}{\sqrt{6}}(1,1,-2).$$

Now

$$\nabla f = \left(z\cos(xz), \frac{1}{y}, x\cos(xz)\right) \quad \Rightarrow \quad \nabla f\big|_{(1,1,\pi)} = (-\pi, 1, -1).$$

Thus

$$\frac{\partial f}{\partial n} = (-\pi, 1, -1) \cdot \frac{1}{\sqrt{6}}(1,1,-2) = \frac{3-\pi}{\sqrt{6}}.$$

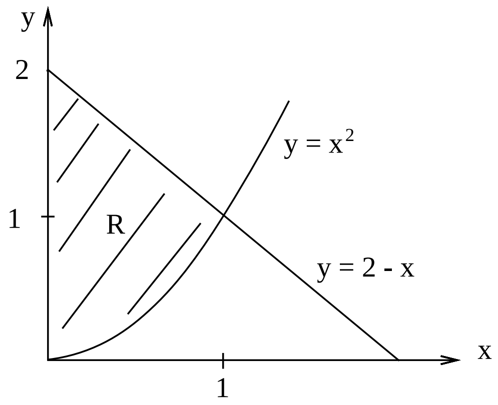
Question 2

(a) The work done is $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$. Let $x = t$ and $y = 3t^3$. Then $\mathbf{r}(t) = (t, 3t^3)$ and $\mathbf{r}'(t) = (1, 9t^2)$. The point $(0,0)$ corresponds to $t = 0$ and $(1,3)$ corresponds to $t = 1$. Also

$$\mathbf{F}(\mathbf{r}(t)) = \big(3(3t^3), -2t^2\big) = \big(9t^3, -2t^2\big).$$

Thus,

$$\begin{aligned}
\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ \mathrm{d}t \\
&= \int_0^1 \big(9t^3, -2t^2\big) \cdot (1, 9t^2)\, dt \\
&= \int_0^1 \big[9t^3 - 18t^4\big]\, dt \\
&= \left[\frac{9}{4}t^4 - \frac{18}{5}t^5\right]_0^1 = -\frac{27}{20}.
\end{aligned}$$



(b)

$$\begin{aligned}\iint_R xy\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_{x^2}^{2-x} xy\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \left[\frac{xy^2}{2}\right]_{x^2}^{2-x} \mathrm{d}x \\ &= \int_0^1 \left[\frac{x^5}{2} - 2x + 2x^2 - \frac{x^3}{2}\right] \mathrm{d}x \\ &= -\frac{3}{8}.\end{aligned}$$

(c) Stokes' Theorem states that $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot \mathrm{d}\mathbf{S}$, where S is an oriented smooth surface bounded by a simple, closed, smooth boundary curve C with positive orientation. Now

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-2z & 2x-z^2 & x+3y \end{vmatrix} = (3-2z, -3, 2).$$

Let C be the circle that bounds the disk S where $z = 2$, $x^2+y^2 \le 1$. Note that the unit normal is $\mathbf{n} = \mathbf{k}$ on S. So, by Stokes' Theorem

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot \mathrm{d}\mathbf{S} = \iint_S (\nabla\times\mathbf{F})\cdot \mathbf{n}\,\mathrm{d}S = 2\iint_S \mathrm{d}S.$$

Now, since $\iint_S \mathrm{d}S$ is just the area of S, which is π, we have

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 2\pi.$$

Alternatively, parameterise S by

$$\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2\,\mathbf{k},$$

for $0 \le \theta \le 2\pi$, $0 \le r \le 1$. Then

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0, 0, r).$$

Also, $(\nabla \times \mathbf{F})(\mathbf{r}(r,\theta)) = (-1, -3, 2)$ and so

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \int_0^1 (\nabla \times \mathbf{F})(\mathbf{r}(r,\theta)) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} \int_0^1 2r\, \mathrm{d}r\, \mathrm{d}\theta = 2\pi.$$

Question 3

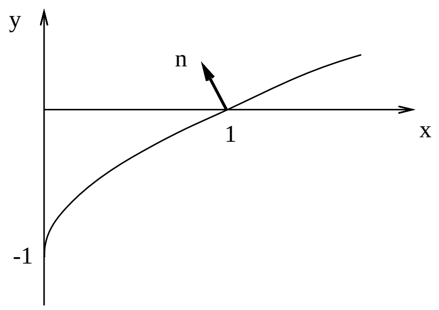
(a) Let $\Omega(x,y) = y - \sqrt{x} + 1$. Then the curve $y - \sqrt{x} + 1 = 0$ is the level curve $\Omega = 0$. Now

$$\nabla\Omega = \left(-\frac{1}{2\sqrt{x}}, 1 \right),$$

and so at the point $(1, 0)$ we have $\nabla\Omega = (-1/2, 1)$. The unit normal is therefore

$$\mathbf{n} = \frac{2}{\sqrt{5}} \left(-\frac{1}{2}, 1 \right).$$

The sketch is shown below.



(b)

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - z^3 & x^2 & -3xz^2 - 1 \end{vmatrix} = \mathbf{i}\,(0-0) - \mathbf{j}\,(-3z^2 + 3z^2) + \mathbf{k}\,(2x - 2x) = \mathbf{0}.$$

Hence $\mathbf{v}$ is conservative and so there exists ϕ such that $\mathbf{v} = \nabla\phi$, i.e.

$$(2xy - z^3, x^2, -3xz^2 - 1) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right),$$

or

$$\text{(i)} \quad \frac{\partial\phi}{\partial x} = 2xy - z^3, \qquad \text{(ii)} \quad \frac{\partial\phi}{\partial y} = x^2, \qquad \text{(iii)} \quad \frac{\partial\phi}{\partial z} = -3xz^2 - 1.$$

(i) $\frac{\partial \phi}{\partial x} = 2xy - z^3$ and so

$$\phi = x^2 y - z^3 x + f(y,z) \qquad \Longrightarrow \qquad \frac{\partial \phi}{\partial y} = x^2 + \frac{\partial f}{\partial y}. \tag{1}$$

(ii) Also $\frac{\partial \phi}{\partial y} = x^2$ and so comparing with (1) we see that $\frac{\partial f}{\partial y} = 0$, or $f = g(z)$ and so

$$\phi = x^2 y - z^3 x + g(z) \qquad \Longrightarrow \qquad \frac{\partial \phi}{\partial z} = -3z^2 x + g'(z). \tag{2}$$

(iii) Finally, $\frac{\partial \phi}{\partial z} = -3xz^2 - 1$ and so comparing with (2) we deduce that $g'(z) = -1$, or $g(z) = -z + c$, where c is an arbitrary constant.

Thus the scalar potential is $\phi(x,y,z) = x^2 y - z^3 x - z + c$. Using the fundamental theorem of line integrals we have

$$\int_C \mathbf{v} \cdot \mathbf{dr} = \int_C \nabla\phi \ \cdot \mathbf{dr} = \phi(1,2,0) - \phi(0,1,0) = (2+c) - (c) = 2.$$

(c) Taylor's series is [From useful information result given at end of paper]:

$$g(x,y,z) = g(x_0,y_0,z_0) + \delta\mathbf{r} \cdot \nabla g\big|_{(x_0,y_0,z_0)} + O\big(|\delta\mathbf{r}|^2\big).$$

Here $(x_0, y_0, y_0) = (1, \pi/2, 1)$ and

$$\delta\mathbf{r} = (h,k,l) = (x - x_0, y - y_0, z - z_0) = (x - 1, y - \pi/2, z - 1),$$

and

$$\nabla g = (\sin y - 2x, \cos y, 1) \qquad \Longrightarrow \qquad \nabla g\big|_{(1,\pi/2,1)} = (-1, 0, 1).$$

Also $g(1, \pi/2, 1) = 1 - 1 + 1 = 1$. Hence

$$g(x,y,z) = 1 + (x-1, y-\pi/2, z-1) \cdot (-1,0,1) + O\big(|\delta\mathbf{r}|^2\big) = 1 - x + z + O\big(|\delta\mathbf{r}|^2\big).$$

If $(x, y) = (1.1, 1.5, 0.9)$ then

$$\delta\mathbf{r} = (0.1, 1.5 - \pi/2, -0.1) \qquad \Rightarrow \qquad |\delta\mathbf{r}|^2 \approx 0.025.$$

Hence

$$g(1.1, 1.5, 0.9) = 1 - 1.1 + 0.9 + 0(0.025) = 0.8 + O(0.025).$$

SECTION B

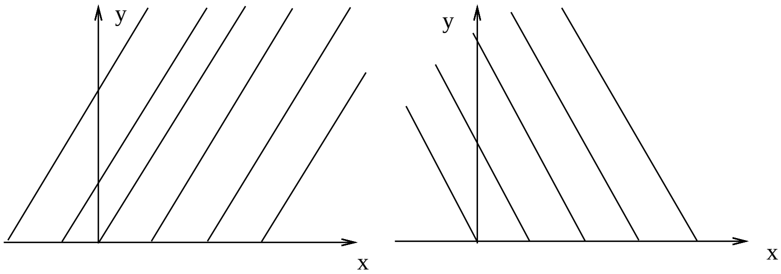
Question 4

(a) An elliptic equation satisfies $B^2 - AC < 0$, a hyperbolic equation satisfies $B^2 - AC > 0$ and a parabolic equation satisfies $B^2 - AC = 0$.

(b) (i) Here $A = 1$, $B = 0$, $C = -3$. Thus $B^2 - AC = 3$ and so the PDE is hyperbolic. The characteristics are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \pm\sqrt{3} \qquad \Longrightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{3}, \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{3},$$

and so (1) $y = \sqrt{3}x + c$ or (2) $y = -\sqrt{3}x + c$.



(1) (2)

(b) (ii) Here $A = 1$, $B = -2$ and $C = 4$. Thus $B^2 - AC = 0$ and so the PDE is parabolic. There is only one set of characteristics which satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B}{A} = -2$$

or $y = -2x + c$, where c is an arbitrary constant. The characteristics are similar to that shown in (2) above.

(b) (iii) Here $A = 1$, $B = 0$ and $C = 3$. Thus $B^2 - AC = -3 < 0$ and so the PDE is elliptic. Thus there are no characteristics.

(c) Now, $u = e^{-ky}\cos(\alpha x)\cos(\beta y)$. Thus

$$\begin{aligned}
u_y &= -ke^{-ky}\cos(\alpha x)\cos(\beta y) - \beta e^{-ky}\cos(\alpha x)\sin(\beta y) \\
&= e^{-ky}\cos(\alpha x)\big[-k\cos(\beta y) - \beta\sin(\beta y)\big] \\
u_{yy} &= -ke^{-ky}\cos(\alpha x)\big[-k\cos(\beta y) - \beta\sin(\beta y)\big] + e^{-ky}\cos(\alpha x)\big[-k\beta\sin(\beta y) - \beta^2\cos(\beta y)\big] \\
&= e^{-ky}\cos(\alpha x)\big[k^2\cos(\beta y) + 2k\beta\sin(\beta y) - \beta^2\cos(\beta y)\big] \\
u_x &= -\alpha e^{-ky}\sin(\alpha x)\cos(\beta y) \\
u_{xx} &= -\alpha^2 e^{-ky}\cos(\alpha x)\cos(\beta y).
\end{aligned}$$

Then

$$\begin{aligned}
& c^2u_{xx} - u_{yy} - 2ku_y \\
= & -c^2\alpha^2 e^{-ky}\cos(\alpha x)\cos(\beta y) - e^{-ky}\cos(\alpha x)\big[k^2\cos(\beta y) + 2k\beta\sin(\beta y) - \beta^2\cos(\beta y)\big] \\
& -2ke^{-ky}\cos(\alpha x)\big[-k\cos(\beta y) - \beta\sin(\beta y)\big] \\
= & \; e^{-ky}\cos(\alpha x)\cos(\beta y)\big[-c^2\alpha^2 + k^2 + \beta^2\big].
\end{aligned}$$

Thus to equal zero requires

$$c^2\alpha^2 = k^2 + \beta^2.$$

(d)

$$xu_x = 1 \quad \Longrightarrow \quad u_x = \frac{1}{x} \quad \Longrightarrow \quad u(x,y) = \ln x + A(y).$$

$$\begin{aligned}
u_y + u^3 = 0 \quad & \Longrightarrow \quad -\frac{\partial u}{\partial y} = u^3 \\
& \Longrightarrow \quad -\frac{\partial u}{u^3} = \partial y \\
& \Longrightarrow \quad \frac{1}{2}u^{-2} = y + B(x) \\
& \Longrightarrow \quad u(x,y) = \frac{1}{\sqrt{2y + 2B(x)}}.
\end{aligned}$$

Question 5

(a) Let $u(x,t) = F(x)G(t)$. Then

$$u_t = F(x)\dot{G}(t) \qquad \& \qquad u_{xx} = F''(x)G(t).$$

Substituting these into $u_t = c^2u_{xx}$ gives

$$F(x)\dot{G}(t) = c^2F''(x)G(t) \qquad \Longrightarrow \qquad \frac{\dot{G}}{c^2G} = \frac{F''}{F} = k,$$

or

$$\dot{G} - c^2kG = 0, \qquad F'' - kF = 0.$$

The boundary conditions $u(0,t) = u(1,t) = 0$ therefore imply $F(0) = F(1) = 0$.

- $k = 0$:

$$F'' = 0 \qquad \Longrightarrow \qquad F(x) = Ax + B.$$

$F(0) = F(l) = 0$ imply that $A = B = 0$ and so there are no non-trivial solutions.

- $k = p^2 > 0$: The solution of $F'' - p^2 F = 0$ is

$$F(x) = Ae^{px} + Be^{-px}.$$

The bc $F(0) = 0$ implies $A + B = 0$ and the bc $F(l) = 0$ implies $Ae^{pl} + Be^{-pl} = 0$. Then $A = B = 0$ and so there are no non-trivial solutions.

- $k = -p^2 < 0$: The solution of $F'' + p^2 F = 0$ is

$$F(x) = A\cos px + B\sin px.$$

The bc $F(0) = 0$ implies $A = 0$ and the bc $F(l) = 0$ implies $B\sin pl = 0$, i.e. $\sin pl = 0$, and so $p = n\pi/l$. Thus

$$F_n(x) = B_n \sin\frac{n\pi x}{l}, \qquad n = 1, 2, \ldots.$$

Now let's look at the ODE for G. We have

$$\dot{G} + \frac{c^2 n^2 \pi^2}{l^2} G_n = 0 \qquad \Longrightarrow \qquad G_n(t) = C_n e^{-\lambda_n^2 t},$$

where $\lambda_n = cn\pi/l$. Hence

$$u_n(x,t) = F_n(x)G_n(t) = B_n \sin\frac{n\pi x}{l}\, C_n e^{-\lambda_n^2 t},$$

and so

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l}\, e^{-\lambda_n^2 t}, \tag{3}$$

where $D_n = B_n C_n$, for $n = 1, 2, \ldots$.

If the IC is $u(x,0) = \sin(2\pi x/l)$ then, using the useful information result given at end of paper,

$$D_n = \frac{2}{l}\int_0^l \sin\frac{2\pi x}{l}\sin\frac{n\pi x}{l}\, \mathrm{d}x.$$

However, it is easier to consider (3) directly. At $t = 0$ it follows that

$$\sin\frac{2\pi x}{l} = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l},$$

and so, by inspection, $D_n = 0$ for all n except $n = 2$ where $D_2 = 1$. Thus

$$u(x,t) = \sin\frac{2\pi x}{l}\, e^{-\lambda_2^2 t}.$$

(b) Let $\hat{u}(x,s) = \int_0^\infty u(x,t)e^{-st}dt$. Then

$$\begin{aligned}
\int_0^\infty u_{tt}(x,t)e^{-st}dt &= \left[u_t(x,t)e^{-st}\right]_0^\infty + s\int_0^\infty u_t(x,t)e^{-st}dt \\
&= -u_t(x,0) + s\left[u(x,t)e^{-st}\right]_0^\infty + s^2\int_0^\infty u(x,t)e^{-st}dt \\
&= -u_t(x,0) - su(x,0) + s^2\hat{u}(x,s) \\
&= s^2\hat{u}(x,s),
\end{aligned}$$

since $u(x,0) = u_t(x,0) = 0$. Hence the PDE becomes

$$\hat{u}_{xx} - \frac{s^2}{c^2}\hat{u} = 0 \qquad \Longrightarrow \qquad \hat{u} = Ae^{-sx/c} + Be^{sx/c}.$$

From the bc $u \to 0$ as $x \to \infty$ it follows that $B = 0$. The bc $u(0,t) = t^2$ implies that

$$\begin{aligned}
\hat{u}(0,s) &= \int_0^\infty t^2 e^{-st}dt = \left[-\frac{1}{s}t^2e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty 2te^{-st}dt \\
&= \frac{2}{s}\left\{\left[-\frac{1}{s}te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt\right\} \\
&= \frac{2}{s^3}.
\end{aligned}$$

Hence $A = 2/s^3$ and so

$$\hat{u} = \frac{2}{s^3}e^{-sx/c} \qquad \Longrightarrow \qquad u(x,t) = H\left(t - \frac{x}{c}\right)\left(t - \frac{x}{c}\right)^2,$$

where $H(\cdot)$ is the Heaviside function (the inverse LT of $1/s^2$ is $t^2/2$) [From useful information result given at end of paper].

Question 6

(a) Use Taylor Series:

$$\begin{aligned}
f(x+2h) &= f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \ldots \\
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots.
\end{aligned}$$

Thus

$$
\begin{aligned}
f'(x) &\cong \frac{-f(x+2h)+4f(x+h)-3f(x)}{2h} \\
&= \frac{1}{2h}\Big\{-\Big[f(x)+2hf'(x)+\frac{(2h)^2}{2!}f''(x)+\frac{(2h)^3}{3!}f'''(x)+\dots\Big] \\
&+4\Big[f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\dots\Big]-3f(x)\Big\} \\
&= \frac{1}{2h}\Big\{2hf'(x)-\frac{2}{3}h^3f'''(x)+\dots\Big\} \\
&= f'(x)-O(h^2),
\end{aligned}
$$

and so the finite difference approximation has $O(h^2)$ accuracy.

(b) Use central differences for the second derivatives, with $\Delta x = \Delta y = h$. Then

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}, \qquad \frac{\partial^2 u}{\partial y^2} \approx \frac{u(x,y+h)-2u(x,y)+u(x,y-h)}{h^2}.$$

Hence the PDE $u_{xx}+u_{yy}=f(x,y)$ is discretised as

$$u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y) \approx h^2 f(x,y). \qquad (4)$$

The square is discretised by a uniform mesh of width $h = 1/N$. The mesh points are the intersections of the lines (x_i, y_j) where $x_i = ih$ and $y_j = jh$ for some $0 \le i,\, j \le N$. Writing the approximation to $u(x_i, y_j)$ as $u_{i,j}$, and similarly for f, equation (4) becomes

$$u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j} = h^2 f_{i,j}, \qquad 1 \le i,\, j \le N-1. \qquad (5)$$

On the boundary we have

$$u_{i,0} = \phi_0(x_i), \qquad u_{i,N} = \phi_1(x_i), \qquad u_{0,j} = \psi_0(y_j), \qquad u_{N,j} = \psi_1(y_j).$$

(c) Using $h = 1/3$ means there are just 4 internal points. Since $f = e^{x+y}$ the difference equation (5) becomes

$$u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j} = h^2 \exp(x_i+y_j), \qquad 1 \le i,\, j \le 2.$$

The boundary conditions are

$$u_{i,0} = x_i(1-x_i), \qquad u_{i,N} = 1, \qquad i = 0,\, 1,\, 2,\, 3$$

$$u_{0,j} = y_j, \qquad u_{N,j} = y_j, \qquad j = 0,\, 1,\, 2,\, 3.$$

Hence the linear system of equations to solve is

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = h^2 \exp(x_1 + y_1)$$

$$u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} - 4u_{2,1} = h^2 \exp(x_2 + y_1)$$

$$u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} - 4u_{1,2} = h^2 \exp(x_1 + y_2)$$

$$u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1} - 4u_{2,2} = h^2 \exp(x_2 + y_2).$$

This determines the unknown interior points $u_{1,1}$, $u_{2,1}$, $u_{1,2}$ and $u_{2,2}$.

(d) Let $u_{i,j} = u(x_i, t_j)$, where $x_i = i\Delta x$, $i = 0, 1, \dots, I$ and $t_j = j\Delta t$, $j = 0, 1, 2, \dots$. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \qquad \frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}.$$

Putting these into $u_t = c^2 u_{xx}$ gives

$$\frac{u_{i,j+1} - u_{i,j}}{c^2 \Delta t} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

or, on rearranging,

$$u_{i,j+1} = \lambda u_{i-1,j} + (1 - 2\lambda) u_{i,j} + \lambda u_{i+1,j}, \tag{6}$$

where $\lambda = c^2 \Delta t / \Delta x^2$ and $i = 1, \dots, I-1$, $j = 0, 1, 2, \dots$.

The initial condition $u(x, 0) = x^2 - 2x + 1$ gives $u_{i,0} = x_i^2 - 2x_i + 1$, $i = 0, 1, \dots, I$. The boundary condition $u(0, t) = 0$ gives $u_{0,j} = 0$, for $j = 0, 1, 2, \dots$.

For the final boundary condition $\partial u(1,t)/\partial x = 0$ we use a central difference to deduce that

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} \equiv \frac{u_{I+1,j} - u_{I-1,j}}{2\Delta x} = 0 \qquad \Longrightarrow \qquad u_{I+1,j} = u_{I-1,j}.$$

Then from (6) with $i = I$ we find that

$$u_{I,j+1} = \lambda u_{I-1,j} + (1 - 2\lambda) u_{I,j} + \lambda u_{I+1,j} = 2\lambda u_{I-1,j} + (1 - 2\lambda) u_{I,j},$$

since $u_{I+1,j} = u_{I-1,j}$.