MA4006: Spring 2010 Exam solutions

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SECTION A

Question 1

(a) Differentiating $\mathbf{r}(t)$ gives

$$\mathbf{r}'(t) = \begin{cases} (1, 0, 1/t), & \text{if } 0.5 \le t \le 1 \\ (0, -2, 1), & \text{if } 1 < t \le 2. \end{cases}$$

Thus

$$|\mathbf{r}'(t)| = \begin{cases} \frac{\sqrt{1+t^2}}{t}, & \text{if } 0.5 \le t \le 1 \\ \sqrt{5}, & \text{if } 1 < t \le 2. \end{cases}$$

and so

$$\hat{\mathbf{t}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \begin{cases} \frac{(t,0,1)}{\sqrt{1+t^2}}, & \text{if } 0.5 \le t \le 1 \\ \frac{(0,-2,1)}{\sqrt{5}}, & \text{if } 1 < t \le 2. \end{cases}$$

As $t \to 1^+$, $\mathbf{r}(t) = (1, 2, 0)$ and $\mathbf{r}'(t) = (0, -2, 1)$ and as $t \to 1^-$, $\mathbf{r}(t) = (1, 2, 0)$ and $\mathbf{r}'(t) = (1, 0, 1)$. Hence the curve is continuous but not smooth at $t = 1$, but smooth elsewhere, and so piecewise smooth.

(b) $\mathbf{r}'(t) = (-3 \sin t, 3 \cos t, 2)$ and so

$$|\mathbf{r}'(t)| = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + 2^2} = \sqrt{13}.$$

Hence

$$s = \int_0^t |\mathbf{r}'(t_0)|\, dt_0 = \int_0^t \sqrt{13}\, dt_0 = \sqrt{13}t,$$

and so the arclength from $t = 0$ to $t = \pi$ is $\sqrt{13}\pi$. Now

$$t = \frac{s}{\sqrt{13}},$$

which means that the intrinsic equation can be written as

$$\mathbf{r}(s) = \left(3 \cos \frac{s}{\sqrt{13}}, 3 \sin \frac{s}{\sqrt{13}}, \frac{2s}{\sqrt{13}}\right).$$

The curvature $\kappa(s)$ is given by

$$\kappa(s) = \left|\mathbf{r}''(s)\right|,$$

and so

$$
\begin{aligned}
\mathbf{r}'(s) &= \left(-\frac{3}{\sqrt{13}}\sin\frac{s}{\sqrt{13}}, \frac{3}{\sqrt{13}}\cos\frac{s}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right) \\
\mathbf{r}''(s) &= \left(-\frac{3}{13}\cos\frac{s}{\sqrt{13}}, -\frac{3}{13}\sin\frac{s}{\sqrt{13}}, 0\right) \\
\Rightarrow \quad \kappa(s) &= \sqrt{\left(-\frac{3}{13}\cos\frac{s}{\sqrt{13}}\right)^2 + \left(-\frac{3}{13}\sin\frac{s}{\sqrt{13}}\right)^2} = \frac{3}{13}.
\end{aligned}
$$

(c) The unit vector in the direction $(1, 2, -2)$ is

$$
\frac{(1,2,-2)}{\sqrt{1^2 + (2)^2 + (-2)^2}} = \frac{1}{3}(1,2,-2).
$$

Now

$$
\nabla f = (2xyz + y^2z + yz^2, x^2z + 2xyz + xz^2, x^2y + xy^2 + 2xyz) \quad \Rightarrow \quad \nabla f\big|_{(1,1,0)} = (0,0,2).
$$

Thus

$$
\frac{\partial f}{\partial n} = (0,0,2) \cdot \frac{1}{3}(1,2,-2) = -\frac{4}{3}.
$$

Question 2

(a) Now

$$\text{curl}\,(\text{grad}\,\omega) = \nabla \times \nabla\omega, \qquad \text{div}\,(\text{curl}\,\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}),$$

and

$$\nabla\omega = (y/z, x/z, -xy/z^2)$$

and so

$$\nabla \times \nabla\omega = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y/z & x/z & -xy/z^2 \end{vmatrix} = \mathbf{i}(-x/z^2 + x/z^2) - \mathbf{j}(-y/z^2 + y/z^2) + \mathbf{k}(1/z - 1/z) = \mathbf{0}.$$

Also

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & \cos y & xy \end{vmatrix} = \mathbf{i}(x - 0) - \mathbf{j}(y - 0) + \mathbf{k}(0 - 0) = (x, -y, 0),$$

and so

$$\nabla \cdot (\nabla \times \mathbf{F}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x, -y, 0) = 0.$$

(b)

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z^2 & 2yz \end{vmatrix} = \mathbf{i}(2z - 2z) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - 0) = \mathbf{0}.$$

Hence $\mathbf{f}$ is conservative and so there exists ϕ such that $\mathbf{f} = \nabla\phi$, i.e.

$$(0, z^2, 2yz) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right),$$

or

$$\text{(i)} \quad \frac{\partial\phi}{\partial y} = z^2, \qquad \text{(ii)} \quad \frac{\partial\phi}{\partial z} = 2yz, \qquad \text{(iii)} \quad \frac{\partial\phi}{\partial x} = 0.$$

(i) $\frac{\partial\phi}{\partial y} = z^2$ and so

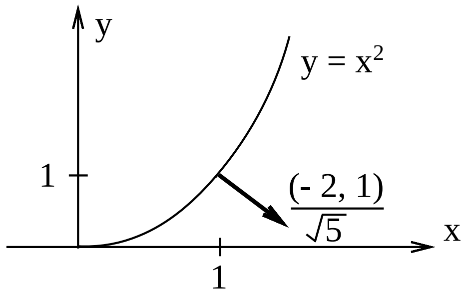
$$\phi = yz^2 + g(x, z) \qquad \Longrightarrow \qquad \frac{\partial\phi}{\partial z} = 2yz + \frac{\partial g}{\partial z}. \tag{1}$$

(ii) Also $\frac{\partial\phi}{\partial z} = 2yz$ and so comparing with (1) we see that $\frac{\partial g}{\partial z} = 0$. So $g = g(x)$ and $\phi = yz^2 + g(x)$.

(iii) Finally, $\frac{\partial\phi}{\partial x} = 0$ which means that $\frac{\partial g}{\partial x} = 0$ and so g is constant.

Thus the scalar potential is $\phi(x, y, z) = yz^2 + c$ where c is an arbitrary constant.

(c) Let $\Omega(x, y) = y - x^2$. Then the curve $y - x^2 = 0$ is the level curve $\Omega = 0$. Now $\nabla\Omega = (-2x, 1)$. So at the point $(1, 1)$ we have $\nabla\Omega = (-2, 1)$. The unit normal is therefore $(-2, 1)/\sqrt{5}$. The sketch is shown below.



(d) Taylor's series is [From useful information result given at end of paper]:

$$f(x, y) = f(x_0, y_0) + \delta\mathbf{r} \cdot \nabla f\big|_{(x_0,y_0)} + O\big(|\delta\mathbf{r}|^2\big).$$

Here $(x_0, y_0) = (0, 1)$ and

$$\delta\mathbf{r} = (h, k) = (x - x_0, y - y_0) = (x, y - 1),$$

and

$$\nabla f = (y\cos(xy) + 2x, x\cos(xy) + 1) \qquad \Longrightarrow \qquad \nabla f\big|_{(0,1)} = (1, 1).$$

Also $f(0, 1) = 1$. Hence

$$f(x, y) = 1 + (x, y - 1) \cdot (1, 1) + O\big(|\delta\mathbf{r}|^2\big) = x + y + O\big(|\delta\mathbf{r}|^2\big).$$

If $(x, y) = (0.1, 1.2)$ then

$$\delta\mathbf{r} = (0.1, 0.2) \qquad \Rightarrow \qquad |\delta\mathbf{r}|^2 = 0.01 + 0.04 = 0.05.$$

Hence

$$f(0.1, 1.2) = 0.1 + 1.2 + O(0.05) = 1.3 + O(0.05).$$

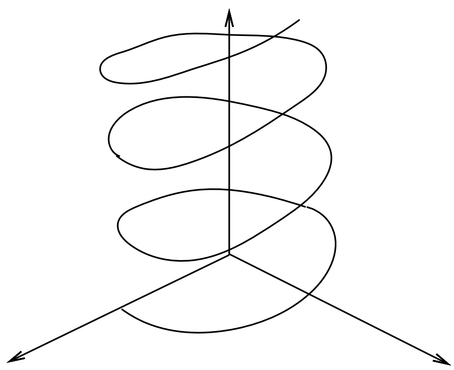
Question 3

(a) The work done is $\int_C \mathbf{F}\cdot d\mathbf{r}$. Here $\mathbf{r}'(t) = (-\sin t, \cos t, 1)$ and

$$\mathbf{F}(\mathbf{r}(t)) = \big(t, \cos t, \sin t\big).$$

The point $(1,0,0)$ corresponds to $t=0$ and the point $(1,0,4\pi)$ corresponds to $t=4\pi$. Thus,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{4\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt \\
&= \int_0^{4\pi} \big(t, \cos t, \sin t\big)\cdot(-\sin t, \cos t, 1)\,dt \\
&= \int_0^{4\pi} \big[-t\sin t + \cos^2 t + \sin t\big]\,dt \\
&= \int_0^{4\pi} \left[-t\sin t + \frac{1}{2} + \frac{1}{2}\cos(2t) + \sin t\right] dt = 4\pi + 2\pi = 6\pi.
\end{aligned}$$



(b) Write the plane as $z = g(x,y) = 1 - x - y$. Then

$$\iint_S \Omega\,dS = \iint_R \Omega(x,y,g(x,y))\sqrt{1+(g_x)^2+(g_y)^2}\,dx\,dy. \tag{2}$$

Here $g_x = -1$ and $g_y = -1$ and $\Omega(x,y,g(x,y)) = x + (1-x-y) = 1-y$. If we integrate first w.r.t. x and second w.r.t. y then the region R is described by $0 \le x \le 1-y$, $0 \le y \le 1$. Thus using (2) we have

$$\begin{aligned}
\iint_S (x+z)\,dS &= \iint_R (1-y)\sqrt{1+(g_x)^2+(g_y)^2}\,dx\,dy \\
&= \int_0^1\int_0^{1-y} (1-y)\sqrt{3}\,dx\,dy \\
&= \sqrt{3}\int_0^1 \big[x(1-y)\big]_0^{1-y}\,dy = \sqrt{3}\int_0^1 (1-y)^2\,dy = \frac{\sqrt{3}}{3}.
\end{aligned}$$

(c) Green's Theorem: Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves. Let $f_1(x,y)$ and $f_2(x,y)$ be functions which are continuous and have continuous partial derivatives $\partial f_1/\partial y$ and $\partial f_2/\partial x$ everywhere in R. Then:

$$\iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \oint_C (f_1 dx + f_2 dy). \tag{3}$$

Consider $\oint_C y^3\,dx - x^3\,dy$. Here $f_1 = y^3$ and $f_2 = -x^3$ and so

$$\frac{\partial f_1}{\partial y} = 3y^2, \qquad \frac{\partial f_2}{\partial x} = -3x^2.$$

Hence using (3) we have

$$\oint_C y^3\,dx - x^3\,dy = \iint_R \left(-3x^2 - 3y^2\right) dx\,dy,$$

where R is a disk of radius 2 centred at the origin. It makes sense to use polar coordinates [From useful information result given at end of paper] and so

$$\begin{aligned}\iint_R \left(-3x^2 - 3y^2\right) dx\,dy &= -3\int_0^{2\pi}\int_0^2 (r^2\cos^2\theta + r^2\sin^2\theta) r\,dr\,d\theta \\ &= -3\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta \\ &= -3\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta \\ &= -12\int_0^{2\pi} d\theta = -24\pi.\end{aligned}$$

SECTION B

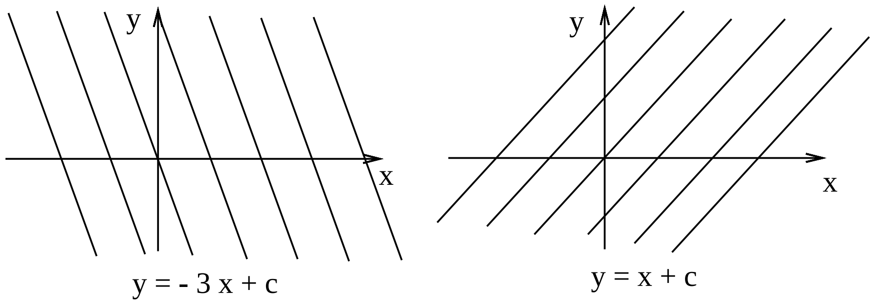
Question 4

(a) A hyperbolic equation satisfies $B^2 - AC > 0$, a parabolic equation satisfies $B^2 - AC = 0$ and an elliptic equation satisfies $B^2 - AC < 0$.

(b) (i) Here $A = 2$, $B = -2$ and $C = -6$. Thus $B^2 - AC = (-2)^2 - 2(-6) = 16 > 0$ and so the PDE is hyperbolic. The characteristics are given by

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{-2 \pm \sqrt{16}}{2} = -1 \pm 2 \qquad \Longrightarrow \qquad \frac{dy}{dx} = -3, \quad \text{or} \quad \frac{dy}{dx} = 1,$$

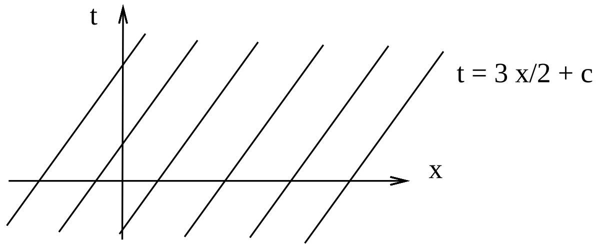
or $y = -3x + c$ and $y = x + c$.



(b) (ii) Here $A = 4$, $B = 6$ and $C = 9$. Thus $B^2 - AC = (6)^2 - 4(9) = 0$ and so the PDE is parabolic. There are only one set of characteristics which satisfy

$$\frac{dt}{dx} = \frac{B}{A} = \frac{3}{2} \qquad \Longrightarrow \qquad t = \frac{3x}{2} + c$$

or $t = 3x/2 + c$.



(b) (iii) Here $A = 1$, $B = 0$ and $C = 1$. Thus $B^2 - AC = -1 < 0$ and so the PDE is elliptic. Thus there are no characteristics.

(c) Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta},$$

and so $u_x = u_\alpha + u_\beta$. This means

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right)\frac{\partial \beta}{\partial x} = \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2},$$

and so $u_{xx} = u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta}$. Also

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial y} = -4\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta},$$

and so $u_y = -4u_\alpha - u_\beta$. This means

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \alpha}\left(-4\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(-4\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right)\frac{\partial \beta}{\partial y} = 16\frac{\partial^2 u}{\partial \alpha^2} + 8\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2},$$

and so $u_{yy} = 16u_{\alpha\alpha} + 8u_{\alpha\beta} + u_{\beta\beta}$. Also

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial \alpha}\left(-4\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(-4\frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}\right)\frac{\partial \beta}{\partial x} = -4\frac{\partial^2 u}{\partial \alpha^2} - 5\frac{\partial^2 u}{\partial \alpha \partial \beta} - \frac{\partial^2 u}{\partial \beta^2},$$

and so $u_{xy} = -4u_{\alpha\alpha} - 5u_{\alpha\beta} - u_{\beta\beta}$. Thus $4u_{xx} + 5u_{xy} + u_{yy} = 0$ becomes

$$4(u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta}) + 5(-4u_{\alpha\alpha} - 5u_{\alpha\beta} - u_{\beta\beta}) + 16u_{\alpha\alpha} + 8u_{\alpha\beta} + u_{\beta\beta} = 0$$

or

$$u_{\alpha\beta} = 0 \qquad \Longrightarrow \qquad u = f(\alpha) + g(\beta).$$

So

$$u(x, y) = f(x - 4y) + g(x - y) \tag{4}$$

$$u_y(x, y) = -4f'(x - 4y) - g'(x - y). \tag{5}$$

From the conditions $u(x, 0) = \cos x$ and $u_y(x, 0) = 0$ we use (4) and (5) to deduce that

$$f(x) + g(x) = \cos x, \qquad -4f'(x) - g'(x) = 0.$$

Thus $-4f(x) - g(x) = k$ where k is an arbitrary constant. Solving these two equations gives

$$f(x) = -\frac{1}{3}\cos x - \frac{1}{3}k, \qquad g(x) = \frac{4}{3}\cos x + \frac{1}{3}k.$$

Finally we have

$$u(x, y) = -\frac{1}{3}\cos(x - 4y) + \frac{4}{3}\cos(x - y).$$

Question 5

(a) We must solve $u_{tt} = c^2 u_{xx}$ on $0 < x < l$, $t > 0$, using the method of separation of variables subject to the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$ and initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Let $u(x,t) = F(x)G(t)$. Then $u_{tt} = F(x)\ddot{G}(t)$ and $u_{xx} = F''(x)G(t)$. Substituting these into $u_{tt} = c^2 u_{xx}$ gives

$$F(x)\ddot{G}(t) = c^2 F''(x)G(t) \qquad \Longrightarrow \qquad \frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k,$$

or

$$\dot{G} - c^2 kG = 0, \qquad F'' - kF = 0.$$

The boundary conditions $u(0,t) = u(l,t) = 0$ therefore imply $F(0) = F(l) = 0$.

- $k = 0$: The solution of $F'' = 0$ is $F(x) = Ax + B$. The bc $F(0) = F(l) = 0$ imply that $A = B = 0$ and so there are no non-trivial solutions.

- $k = p^2 > 0$: The solution of $F'' - p^2 F = 0$ is $F(x) = Ae^{px} + Be^{-px}$. The bc $F(0) = 0$ implies $A + B = 0$ and the bc $F(l) = 0$ implies $Ae^{pl} + Be^{-pl} = 0$. Then $A = B = 0$ and so there are no non-trivial solutions.

- $k = -p^2 < 0$: The solution of $F'' + p^2 F = 0$ is $F(x) = A\cos px + B\sin px$. The bc $F(0) = 0$ implies $A = 0$ and the bc $F(l) = 0$ implies $B\sin(pl) = 0$, and so $pl = n\pi$. Thus

$$F_n(x) = B_n \sin\frac{n\pi x}{l}.$$

Now solve the ODE for G. We have

$$\ddot{G} + \frac{c^2 n^2 \pi^2}{l^2} G_n = 0 \qquad \Longrightarrow \qquad G_n(t) = \bar{C}_n \cos(\lambda_n t) + \bar{D}_n \sin(\lambda_n t),$$

where $\lambda_n = cn\pi/l$. Hence

$$u_n(x,t) = F_n(x)G_n(t) = \left[C_n \cos(\lambda_n t) + D_n \sin(\lambda_n t)\right] \sin\frac{n\pi x}{l},$$

where $C_n = \bar{C}_n B_n$ and $D_n = \bar{D}_n B_n$. This gives

$$u(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos(\lambda_n t) + D_n \sin(\lambda_n t)\right] \sin\frac{n\pi x}{l}.$$

Finally, to find C_n and D_n we use the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$. Now

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \big[-\lambda_n C_n \sin(\lambda_n t) + \lambda_n D_n \cos(\lambda_n t) \big)\big] \sin \frac{n\pi x}{l}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l}, \qquad 0 = \sum_{n=1}^{\infty} \lambda_n D_n \sin \frac{n\pi x}{l},$$

and so

$$C_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \qquad D_n = 0.$$

[From useful information result given at end of paper].

(b) Let $\hat{u}(x,s) = \int_0^\infty u(x,t) e^{-st} dt$. Then

$$\begin{aligned} \int_0^\infty u_{tt}(x,t) e^{-st} dt &= \big[u_t(x,t)e^{-st}\big]_0^\infty + s \int_0^\infty u_t(x,t) e^{-st} dt \\ &= -u_t(x,0) + s\big[u(x,t)e^{-st}\big]_0^\infty + s^2 \int_0^\infty u(x,t) e^{-st} dt \\ &= -u_t(x,0) - su(x,0) + s^2 \hat{u}(x,s) \\ &= s^2 \hat{u}(x,s), \end{aligned}$$

since $u(x,0) = u_t(x,0) = 0$. Hence the PDE becomes (with $c = 1$)

$$\hat{u}_{xx} - s^2 \hat{u} = 0 \qquad \Longrightarrow \qquad \hat{u} = Ae^{-sx} + Be^{sx}.$$

From the bc $u \to 0$ as $x \to \infty$ it follows that $B = 0$. The bc $u(0,t) = e^{-t}$ implies that

$$\hat{u}(0,s) = \int_0^\infty e^{-(s+1)t} dt = \frac{1}{1+s}.$$

Hence $A = 1/(1+s)$ and so

$$\hat{u} = \frac{1}{1+s} e^{-sx} \qquad \Longrightarrow \qquad u(x,t) = H(t-x) e^{-(t-x)},$$

where $H(\cdot)$ is the Heaviside function.

(c) Now

$$\begin{aligned} u_t &= 4\cos 4t \cos x, \qquad & u_{tt} &= -16 \sin 4t \cos x \\ u_x &= -\sin 4t \sin x, \qquad & u_{xx} &= -\sin 4t \cos x. \end{aligned}$$

Hence

$$u_{tt} - c^2 u_{xx} = -16 \sin 4t \cos x + c^2 \sin 4t \cos x = 0$$

if $c^2 = 16$, i.e. $c = \pm 4$.

Question 6

(a) Use Taylor Series:

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \dots$$
$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots.$$

Thus

$$\begin{aligned} f'(x) &\cong \frac{-f(x+2h)+4f(x+h)-3f(x)}{2h} \\ &= \frac{1}{2h}\Big\{-\Big[f(x)+2hf'(x)+\frac{(2h)^2}{2!}f''(x)+\frac{(2h)^3}{3!}f'''(x)+\dots\Big] \\ &+4\Big[f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\dots\Big]-3f(x)\Big\} \\ &= \frac{1}{2h}\Big\{2hf'(x)-\frac{2}{3}h^3f'''(x)+\dots\Big\} \\ &= f'(x)-O(h^2), \end{aligned}$$

and so the finite difference approximation has $O(h^2)$ accuracy.

(b) Use central differences for the second derivatives, with $\Delta x = \Delta y = h$. Then

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}, \qquad \frac{\partial^2 u}{\partial y^2} \approx \frac{u(x,y+h)-2u(x,y)+u(x,y-h)}{h^2}.$$

Hence the PDE $u_{xx}+u_{yy}=f(x,y)$ is discretised as

$$u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y) \approx h^2 f(x,y). \tag{6}$$

The square is discretised by a uniform mesh of width $h = 1/N$. The mesh points are the intersections of the lines (x_i, y_j) where $x_i = ih$ and $y_j = jh$ for some $0 \le i,\, j \le N$. Writing the approximation to $u(x_i, y_j)$ as $u_{i,j}$, and similarly for f, equation (6) becomes

$$u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j} = h^2 f_{i,j}, \qquad 1 \le i,\, j \le N-1. \tag{7}$$

On the boundary we have

$$u_{i,0} = \phi_0(x_i), \qquad u_{i,N} = \phi_1(x_i), \qquad u_{0,j} = \psi_0(y_j), \qquad u_{N,j} = \psi_1(y_j).$$

(c) Using $h = 1/3$ means there are just 4 internal points. Since $f = \sin(x+y)$ the difference equation (7) becomes

$$u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j} = h^2\sin(x_i+y_j), \qquad 1 \le i,\, j \le 2.$$

The boundary conditions are

$$u_{i,0} = 1, \qquad u_{i,N} = x_i(1 - x_i), \qquad i = 0,\, 1,\, 2,\, 3$$

$$u_{0,j} = 1 - y_j, \qquad u_{N,j} = 1 - y_j^2, \qquad j = 0,\, 1,\, 2,\, 3.$$

Hence the linear system of equations to solve is

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = h^2 \sin(x_1 + y_1)$$

$$u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} - 4u_{2,1} = h^2 \sin(x_2 + y_1)$$

$$u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} - 4u_{1,2} = h^2 \sin(x_1 + y_2)$$

$$u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1} - 4u_{2,2} = h^2 \sin(x_2 + y_2).$$

This determines the unknown interior points $u_{1,1}$, $u_{2,1}$, $u_{1,2}$ and $u_{2,2}$.

(d) Let $u_{i,j} = u(x_i, t_j)$, where $x_i = i\Delta x$, $i = 0, 1, \ldots, I$ and $t_j = j\Delta t$, $j = 0, 1, 2, \ldots$. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \qquad \frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}.$$

Putting these into $u_t = c^2 u_{xx}$ gives

$$\frac{u_{i,j+1} - u_{i,j}}{c^2 \Delta t} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

or, on rearranging,

$$u_{i,j+1} = \lambda u_{i-1,j} + (1 - 2\lambda) u_{i,j} + \lambda u_{i+1,j}, \tag{8}$$

where $\lambda = c^2 \Delta t / \Delta x^2$ and $i = 1, \ldots, I-1$, $j = 0, 1, 2, \ldots$.

The initial condition $u(x, 0) = x^2$ gives $u_{i,0} = x_i^2$, $i = 0, 1, \ldots, I$. The boundary condition $u(1, t) = 1$ gives $u_{I,j} = 1$, for $j = 0, 1, 2, \ldots$.

For the final boundary condition $\partial u(0, t)/\partial x = 0$ we use a central difference to deduce that

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} \equiv \frac{u_{-1,j} - u_{1,j}}{2\Delta x} = 0 \qquad \Longrightarrow \qquad u_{-1,j} = u_{1,j}.$$

Then from (8) with $i = 0$ we find that

$$u_{0,j+1} = \lambda u_{-1,j} + (1 - 2\lambda) u_{0,j} + \lambda u_{1,j} = (1 - 2\lambda) u_{0,j} + 2\lambda u_{1,j},$$

since $u_{-1,j} = u_{1,j}$.