MA4006: Spring 2009 Exam solutions

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SECTION A

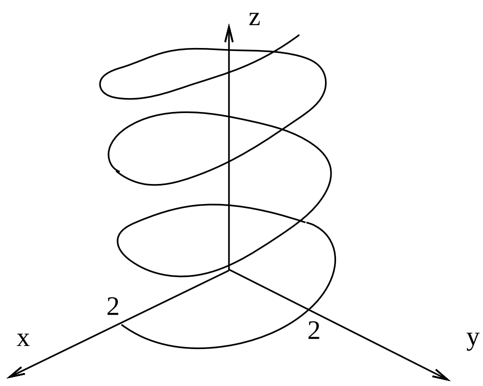
Question 1

(a) Since $\hat{\mathbf{u}}$ is a unit vector it follows that

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = 1 \quad \Rightarrow \quad \frac{d\hat{\mathbf{u}}}{dt} \cdot \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \frac{d\hat{\mathbf{u}}}{dt} = 0 \quad \Rightarrow \quad 2\hat{\mathbf{u}} \cdot \frac{d\hat{\mathbf{u}}}{dt} = 0,$$

which implies that $\hat{\mathbf{u}}$ is perpendicular to $\frac{d\hat{\mathbf{u}}}{dt}$.

(b) (i)



(b) (ii) Since $\mathbf{r} = (2\cos 3t, 2\sin 3t, 4t)$, we have

$$\mathbf{v} = \mathbf{r}' = (-6\sin 3t, 6\cos 3t, 4), \qquad \mathbf{a} = \mathbf{v}' = (-18\cos 3t, -18\sin 3t, 0).$$

Thus

$$|\mathbf{v}| = \sqrt{40}, \qquad |\mathbf{a}| = 18.$$

(b) (iii) To show they are perpendicular we must show that $\mathbf{v} \cdot \mathbf{a} = 0$. So

$$\mathbf{v} \cdot \mathbf{a} = (-6\sin 3t, 6\cos 3t, 4) \cdot (-18\cos 3t, -18\sin 3t, 0) = 108\sin 3t\cos 3t - 108\sin 3t\cos 3t = 0.$$

This could not have been deduced from (a) because $\mathbf{v}$ is not a unit vector.

(b) (iv) Arclength is

$$s = \int_0^t |\mathbf{r}'(t_0)|\, dt_0 = \int_0^t |\mathbf{v}(t_0)|\, dt_0 = \int_0^t \sqrt{40}\, dt_0 = \sqrt{40}\, t.$$

(b) (v) From part (iv) we have $t = s/\sqrt{40}$. Then the intrinsic equation can be written as

$$\mathbf{r}(s) = \left(2\cos\frac{3s}{\sqrt{40}}, 2\sin\frac{3s}{\sqrt{40}}, \frac{4s}{\sqrt{40}}\right).$$

The curvature $\kappa(s)$ is given by

$$\kappa(s) = \left|\mathbf{r}''(s)\right|,$$

and so

$$\begin{aligned}
\mathbf{r}'(s) &= \left(-\frac{6}{\sqrt{40}}\sin\frac{3s}{\sqrt{40}}, \frac{6}{\sqrt{40}}\cos\frac{3s}{\sqrt{40}}, \frac{4}{\sqrt{40}}\right) \\
\mathbf{r}''(s) &= \left(-\frac{18}{40}\cos\frac{3s}{\sqrt{40}}, -\frac{18}{40}\sin\frac{3s}{\sqrt{40}}, 0\right) \\
\Rightarrow \quad \kappa(s) &= \sqrt{\left(-\frac{18}{40}\cos\frac{3s}{\sqrt{40}}\right)^2 + \left(-\frac{18}{40}\sin\frac{3s}{\sqrt{40}}\right)^2} = \frac{18}{40} = \frac{9}{20}.
\end{aligned}$$

(c) The unit vector in the direction $(1, -2, 0)$ is

$$\frac{(1,-2,0)}{\sqrt{1^2 + (-2)^2}} = \frac{1}{\sqrt{5}}(1,-2,0).$$

Now

$$\nabla f = (4x, 6y, 2z) \quad \Rightarrow \quad \nabla f\big|_{(2,1,3)} = (8,6,6).$$

Thus

$$\frac{\partial f}{\partial n} = (8,6,6)\cdot\frac{1}{\sqrt{5}}(1,-2,0) = -\frac{4}{\sqrt{5}}.$$

Question 2

(a)

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3\cos x - 4x^3z & 2z^3y\sin x & 3y^2z^2\sin x - x^4 \end{vmatrix}$$

$$= \mathbf{i}\big[6yz^2\sin x - 6yz^2\sin x\big] - \mathbf{j}\big[3y^2z^2\cos x - 4x^3 - (3y^2z^2\cos x - 4x^3)\big]$$

$$+ \mathbf{k}\big[2z^3y\cos x - 2z^3y\cos x\big]$$

$$= \mathbf{0}.$$

Hence $\mathbf{f}$ is conservative and so there exists ϕ such that $\mathbf{f} = \nabla\phi$, i.e.

$$(y^2z^3\cos x - 4x^3z, 2z^3y\sin x, 3y^2z^2\sin x - x^4) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right),$$

or

$$\text{(i)} \quad \frac{\partial\phi}{\partial x} = y^2z^3\cos x - 4x^3z, \qquad \text{(ii)} \quad \frac{\partial\phi}{\partial y} = 2z^3y\sin x, \qquad \text{(iii)} \quad \frac{\partial\phi}{\partial z} = 3y^2z^2\sin x - x^4.$$

(i) $\frac{\partial\phi}{\partial x} = y^2z^3\cos x - 4x^3z$ and so

$$\phi = y^2z^3\sin x - x^4z + g(y,z) \qquad \Longrightarrow \qquad \frac{\partial\phi}{\partial y} = 2yz^3\sin x + \frac{\partial g}{\partial y}. \tag{1}$$

(ii) Also $\frac{\partial\phi}{\partial y} = 2z^3y\sin x$ and so comparing with (1) we see that $\frac{\partial g}{\partial y} = 0$. So $g = g(z)$, $\phi = y^2z^3\sin x - x^4z + g(z)$ and $\frac{\partial\phi}{\partial z} = 3y^2z^2\sin x - x^4 + g'(z)$.

(iii) Since $\frac{\partial\phi}{\partial z} = 3y^2z^2\sin x - x^4$ we therefore deduce that $g'(z) = 0$ and so g is constant.

Thus the scalar potential is $\phi(x,y,z) = y^2z^3\sin x - x^4 + c$ where c is an arbitrary constant.

(b) The level surfaces of $\Omega(x,y,z)$ are those surfaces defined by

$$\Omega(x,y,z) = c, \qquad \text{where } c \text{ is constant.}$$

$\nabla\Omega$ is normal to the level surface $\Omega(x,y,z) = c$. Consider $x^2 + 2y^2 + z^2 = 7$. We can write this as $\Omega(x,y,z) = 0$ where

$$\Omega(x,y,z) = x^2 + 2y^2 + z^2 - 7.$$

Then

$$\nabla\Omega = 2x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k}.$$

Thus at the point $(1, 1, 2)$:

$$\nabla\Omega = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k},$$

and so the unit normal is

$$\frac{1}{\sqrt{2^2 + 4^2 + 4^2}}\big(2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}\big) = \frac{1}{3}(1, 2, 2).$$

(c) Taylor's series is [From useful information result given at end of paper]:

$$f(x, y) = f(x_0, y_0) + \delta\mathbf{r} \cdot \nabla f\big|_{(x_0,y_0)} + O\big(|\delta\mathbf{r}|^2\big).$$

Here $(x_0, y_0) = (2, 0)$ and

$$\delta\mathbf{r} = (h, k) = (x - x_0, y - y_0) = (x - 2, y),$$

and

$$\nabla f = (-y\sin(xy) + ye^{xy}, -x\sin(xy) + xe^{xy}) \qquad \Longrightarrow \qquad \nabla f\big|_{(2,0)} = (0, 2).$$

Also $f(2, 0) = 2$. Hence

$$f(x, y) = 2 + (x - 2, y) \cdot (0, 2) + O\big(|\delta\mathbf{r}|^2\big) = 2 + 2y + O\big(|\delta\mathbf{r}|^2\big).$$

If $(x, y) = (1.5, 0.3)$ then

$$\delta\mathbf{r} = (0.5, 0.3) \qquad \Rightarrow \qquad |\delta\mathbf{r}|^2 = 0.05 + 0.09 = 0.14.$$

Hence

$$f(1.5, 0.3) = 2 + 2(0.3) + O(0.14) = 2.6 + O(0.14).$$

Question 3

(a) The work done is $\int_C \mathbf{F}\cdot d\mathbf{r}$. Let $x = t$ and $y = 2t^2$. Then $\mathbf{r}(t) = (t, 2t^2)$ and $\mathbf{r}'(t) = (1, 4t)$. The point $(0, 0)$ corresponds to $t = 0$ and $(1, 2)$ corresponds to $t = 1$. Also

$$\mathbf{F}(\mathbf{r}(t)) = \big(3(t)(2t^2), -(2t^2)^2\big) = \big(6t^3, -4t^4\big).$$

Thus,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt \\ &= \int_0^1 \big(6t^3, -4t^4\big)\cdot(1, 4t)\,dt \\ &= \int_0^1 \big[6t^3 - 16t^5\big]\,dt \\ &= \left[\frac{3}{2}t^4 - \frac{8}{3}t^6\right]_0^1 = -\frac{7}{6}.\end{aligned}$$

(b) Use polar co-ordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad 0 \le \theta \le 2\pi, \quad 0 \le r \le 1.$$

Then

$$\iint_R e^{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 e^{r^2} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 \frac{d}{dr}\big(e^{r^2}\big)\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\big[e^{r^2}\big]_0^1\,d\theta = \pi(e-1).$$

(c) The divergence theorem: Consider a **closed** region V bounded by a piecewise smooth closed surface S. If the vector $\mathbf{f}$ is defined and continuously differentiable throughout V then

$$\iint_S \mathbf{f}\cdot\mathbf{dS} = \iiint_V \nabla\cdot\mathbf{f}\,dV.$$

Now

$$\nabla\cdot\mathbf{f} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(xy, -\frac{1}{2}y^2, z\right) = y - y + 1 = 1.$$

Use cylindrical co-ordinates [From useful information result given at end of paper]:

$$\mathbf{r} = (r\cos\theta, r\sin\theta, z), \qquad 0 \le \theta \le 2\pi, \quad 1 \le z \le 4.$$

Note that when $z = 1$ we have $x^2 + y^2 = 1$ and so $0 \le r \le 1$. Then

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r, \qquad (\nabla\cdot\mathbf{f})(\mathbf{r}(r, \theta, z)) = 1.$$

Thus

$$\iiint_V \nabla\cdot\mathbf{f}\,dV = \int_1^4\int_0^{2\pi}\int_0^1 r\,dr\,d\theta\,dz = \frac{1}{2}\int_1^4\int_0^{2\pi} d\theta\,dz = \pi\int_1^4 dz = 3\pi.$$

SECTION B

Question 4

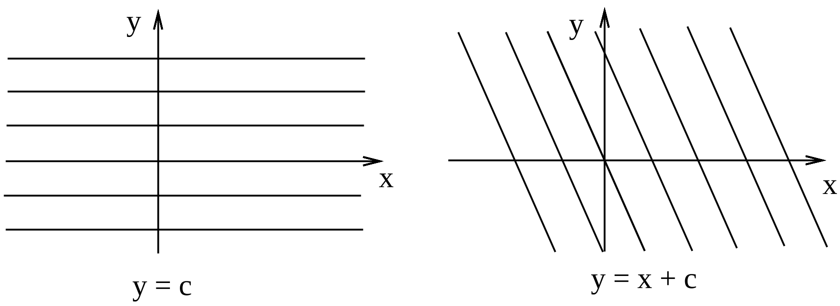
(a) A hyperbolic equation satisfies $B^2 - AC > 0$, a parabolic equation satisfies $B^2 - AC = 0$ and an elliptic equation satisfies $B^2 - AC < 0$.

(b) (i) Here $A = 1$, $B = 1/2$ and $C = 1$. Thus $B^2 - AC = 1/4 - 1 = -3/4 < 0$ and so the PDE is elliptic. Thus there are no characteristics.

(b) (ii) Here $A = 1$, $B = 3/2$ and $C = 0$. Thus $B^2 - AC = 9/4 - 0 = 9/4 > 0$. The characteristics are given by

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = -3/2 \pm \sqrt{9/4} = -3/2 \pm 3/2 \qquad \Longrightarrow \qquad \frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -3,$$

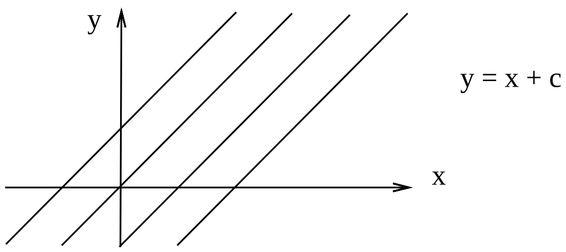
or $y = c$ and $y = -3x + c$.



(b) (iii) Here $A = 1$, $B = 1$ and $C = 1$. Thus $B^2 - AC = 1 - 1 = 0$ and so the PDE is parabolic. There are only one set of characteristics which satisfy

$$\frac{dy}{dx} = \frac{B}{A} = 1 \qquad \Longrightarrow \qquad y = x + c$$

or $y = x + c$.



(c) (i) Let $\xi = x + ay$ and $\eta = x + by$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

and so $u_x = u_\xi + u_\eta$. Also

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

or $u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$. Now

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta},$$

or $u_y = au_\xi + bu_\eta$. Also

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \xi}\left(a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left(a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial y} = a^2\frac{\partial^2 u}{\partial \xi^2} + 2ab\frac{\partial^2 u}{\partial \xi \partial \eta} + b^2\frac{\partial^2 u}{\partial \eta^2},$$

or $u_{yy} = a^2 u_{\xi\xi} + 2abu_{\xi\eta} + b^2 u_{\eta\eta}$. Finally

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial \xi}\left(a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = a\frac{\partial^2 u}{\partial \xi^2} + (a+b)\frac{\partial^2 u}{\partial \xi \partial \eta} + b\frac{\partial^2 u}{\partial \eta^2},$$

or $u_{xy} = au_{\xi\xi} + (a+b)u_{\xi\eta} + bu_{\eta\eta}$. Thus $u_{xx} - 2u_{xy} - 8u_{yy} = 0$ becomes

$$\begin{aligned} & u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - 2\big(au_{\xi\xi} + (a+b)u_{\xi\eta} + bu_{\eta\eta}\big) - 8\big(a^2u_{\xi\xi} + 2abu_{\xi\eta} + b^2u_{\eta\eta}\big) = 0 \\ \Longrightarrow \quad & (1 - 2a - 8a^2)u_{\xi\xi} + 2(1 - a - b - 8ab)u_{\xi\eta} + (1 - 2b - 8b^2)u_{\eta\eta} = 0. \end{aligned}$$

We want $u_{\xi\eta} = 0$ and so we choose $1 - 2a - 8a^2 = 0$ and $1 - 2b - 8b^2 = 0$. If we take the two roots we have $a = 1/4$ and $b = -1/2$. Then $(9/4)u_{\xi\eta} = 0$ or $u_{\xi\eta} = 0$ and so

$$u_\xi = \phi(\xi) \qquad \Longrightarrow \qquad u = \int \phi(\xi)\,d\xi + g(\eta) = f(\xi) + g(\eta).$$

Finally this gives

$$u(x,y) = f(x + y/4) + g(x - y/2).$$

(c) (ii) Now $u_y(x,y) = f'(x+y/4)/4 - g'(x-y/2)/2$. Thus $u(x,0) = \cos(4x)$ and $u_y(x,0) = -\sin(4x)$ implies

$$\cos(4x) = f(x) + g(x), \qquad -\sin(4x) = f'(x)/4 - g'(x)/2.$$

Integrating the second gives $\cos(4x) = f(x) - 2g(x) + c$. Combining with the first implies $g(x) = c/3$. Then $f(x) = \cos(4x) - c/3$ and we deduce that $u(x,y) = \cos(4x + y)$.

Question 5

(a) Let $u(x,t) = F(x)G(t)$. Then

$$u_t = F(x)\dot{G}(t) \qquad \& \qquad u_{xx} = F''(x)G(t).$$

Substituting these into $u_t = c^2 u_{xx}$ gives

$$F(x)\dot{G}(t) = c^2 F''(x)G(t) \qquad \Longrightarrow \qquad \frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k,$$

or

$$\dot{G} - c^2 kG = 0, \qquad F'' - kF = 0.$$

The boundary conditions $u_x(0,t) = u_x(1,t) = 0$ therefore imply $F'(0) = F'(1) = 0$.

- $k = 0$:

$$F'' = 0 \qquad \Longrightarrow \qquad F(x) = Ax + B.$$

  $F'(0) = F'(l) = 0$ imply that $A = 0$. Thus $F = B = \text{const}$.

- $k = p^2 > 0$: The solution of $F'' - p^2 F = 0$ is

$$F(x) = Ae^{px} + Be^{-px}.$$

  The bc $F'(0) = 0$ implies $Ap - Bp = 0$ and the bc $F'(l) = 0$ implies $Ape^{pl} - Bpe^{-pl} = 0$. Then $A = B = 0$ and so there are no non-trivial solutions.

- $k = -p^2 < 0$: The solution of $F'' + p^2 F = 0$ is

$$F(x) = A\cos px + B\sin px \qquad \Rightarrow \qquad F'(x) = -pA\sin px + Bp\cos px.$$

  The bc $F'(0) = 0$ implies $B = 0$ and the bc $F'(l) = 0$ implies $-pA\sin pl = 0$, i.e. $\sin pl = 0$, and so $p = n\pi/l$. Thus

$$F_n(x) = A_n \cos\frac{n\pi x}{l}, \qquad n = 0, 1, 2, \ldots.$$

  Note that $n = 0$ includes the constant solution when $k = 0$.

Now let's look at the ODE for G. We have

$$\dot{G} + \frac{c^2 n^2 \pi^2}{l^2} G_n = 0 \qquad \Longrightarrow \qquad G_n(t) = C_n e^{-\lambda_n^2 t},$$

where $\lambda_n = cn\pi/l$. Hence

$$u_n(x,t) = F_n(x)G_n(t) = A_n \cos\frac{n\pi x}{l}\, C_n e^{-\lambda_n^2 t},$$

and so

$$u(x,t) = \frac{D_0}{2} + \sum_{n=1}^{\infty} D_n \cos\frac{n\pi x}{l}\, e^{-\lambda_n^2 t},$$

where $D_0 = 2A_0C_0$ and $D_n = A_nC_n$, for $n = 1, 2, \ldots$ [It is convenient to keep the constant term in this form for using the Fourier cosine series]. Finally, to find D_n we use the initial condition $u(x,0) = u_0x/l$. Then

$$\frac{u_0 x}{l} = \frac{D_0}{2} + \sum_{n=1}^{\infty} D_n \cos\frac{n\pi x}{l}\ ,$$

and so [from useful information result given at end of paper]

$$\begin{aligned} D_n &= \frac{2}{l}\int_0^l \frac{u_0 x}{l}\cos\frac{n\pi x}{l}\,dx \\ &= \frac{2u_0}{n^2\pi^2}\big[\cos n\pi - 1\big]. \end{aligned}$$

(b) Let $\hat{u}(x,s) = \int_0^\infty u(x,t)e^{-st}dt$. Then

$$\begin{aligned} \int_0^\infty u_t(x,t)e^{-st}dt &= \big[u(x,t)e^{-st}\big]_0^\infty + s\int_0^\infty u(x,t)e^{-st}dt \\ &= -u(x,0) + s\int_0^\infty u(x,t)e^{-st}dt \\ &= s\hat{u}(x,s), \end{aligned}$$

since $u(x,0) = 0$. Hence the PDE becomes

$$c^2\hat{u}_{xx} - s\hat{u} = 0 \qquad \Longrightarrow \qquad \hat{u}(x,s) = Ae^{-\sqrt{s}x/c} + Be^{\sqrt{s}x/c}.$$

The BC $u_x(0,t) = -1$ implies that

$$\hat{u}_x(0,s) = \int_0^\infty u_x(0,t)e^{-st}dt = -\int_0^\infty e^{-st}dt = -\frac{1}{s}.$$

From the bc $u \to 0$ as $x \to \infty$ it follows that $\hat{u} \to 0$ as $x \to \infty$ and so $B = 0$. Then

$$\hat{u}_x(x,s) = -\frac{A\sqrt{s}}{c}e^{-\sqrt{s}x/c},$$

and so $\hat{u}_x(0,s) = -1/s$ implies that $A = c/s^{3/2}$. Thus

$$\hat{u}(x,s) = \frac{c}{s^{3/2}}e^{-\sqrt{s}x/c} \qquad \Longrightarrow \qquad u(x,t) = c\left[2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{x^2}{4c^2t}\right) - \frac{x}{c}\mathrm{erfc}\left(\frac{x}{2c\sqrt{t}}\right)\right].$$

[From useful information result given at end of paper].

Question 6

(a) Use Taylor Series:

$$\begin{aligned} f(x+3h) &= f(x) + 3hf'(x) + \frac{(3h)^2}{2!}f''(x) + \frac{(3h)^3}{3!}f'''(x) + \dots \\ f(x+2h) &= f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \dots \\ f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots . \end{aligned}$$

Thus

$$\begin{aligned} f'(x) &\cong \frac{2f(x+3h) - 9f(x+2h) + 18f(x+h) - 11f(x)}{6h} \\ &= \frac{1}{2h}\Big\{2\Big[f(x) + 3hf'(x) + \frac{(3h)^2}{2!}f''(x) + \frac{(3h)^3}{3!}f'''(x) + \dots\Big] \\ &-9\Big[f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \dots\Big] \\ &+18\Big[f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots\Big] - 11f(x)\Big\} \\ &= \frac{1}{6h}\left\{6hf'(x) + O(h^4) + \dots\right\} = f'(x) + O(h^3), \end{aligned}$$

and so the finite difference approximation has $O(h^3)$ accuracy.

(b) Use central differences for the second derivatives, with $\Delta x = \Delta y = h$. Then

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}, \qquad \frac{\partial^2 u}{\partial y^2} \approx \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}.$$

Hence the PDE $u_{xx} + u_{yy} = f(x,y)$ is discretised as

$$u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y) \approx h^2 f(x,y). \qquad (2)$$

The square is discretised by a uniform mesh of width $h = 1/N$. The mesh points are the intersections of the lines (x_i, y_j) where $x_i = ih$ and $y_j = jh$ for some $0 \leq i,\, j \leq N$. Writing the approximation to $u(x_i, y_j)$ as $u_{i,j}$, and similarly for f, equation (2) becomes

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j}, \qquad 1 \leq i,\, j \leq N-1. \qquad (3)$$

On the boundary we have

$$u_{i,0} = g_{i,0}, \qquad u_{i,N} = g_{i,N}, \qquad u_{0,j} = g_{0,j}, \qquad u_{N,j} = g_{N,j}.$$

(c) Using $h = 1/3$ means there are just 4 internal points. Since $f = 0$ the difference equation (3) becomes

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0, \qquad 1 \le i,\, j \le 2.$$

The boundary conditions are

$$u_{i,0} = x_i(1 - x_i), \qquad u_{i,N} = 1, \qquad i = 0,\, 1,\, 2,\, 3$$

$$u_{0,j} = y_j, \qquad u_{N,j} = y_j, \qquad j = 0,\, 1,\, 2,\, 3.$$

Hence the linear system of equations to solve is

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = 0$$

$$u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} - 4u_{2,1} = 0$$

$$u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} - 4u_{1,2} = 0$$

$$u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1} - 4u_{2,2} = 0.$$

This determines the unknown interior points $u_{1,1}$, $u_{2,1}$, $u_{1,2}$ and $u_{2,2}$.

(d) Let $u_{i,j} = u(x_i, t_j)$, where $x_i = i\Delta x$, $i = 0, 1, \ldots, I$ and $t_j = j\Delta t$, $j = 0, 1, 2, \ldots$. Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \qquad \frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2}.$$

Putting these into $u_{tt} = c^2 u_{xx}$ gives

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{c^2\Delta t^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

or, on rearranging,

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \lambda^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}), \tag{4}$$

where $\lambda = c\Delta t/\Delta x$, $\Delta x = 1/I$ and $i = 1, \ldots, I-1$, $j = 1, 2, \ldots$. The boundary condition $u(0,t) = 0$ implies $u_{0,j} = 0$. The initial condition $u(x,0) = xe^{-5(x-1)^2}$ implies $u_{i,0} = x_i e^{-5(x_i-1)^2}$, $i = 0, 1, \ldots, I$. For the initial condition $u_t(x,0) = 0$ we use a central difference to deduce that

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} \equiv \frac{u_{i,1} - u_{i,-1}}{2\Delta t} = 0 \qquad \Longrightarrow \qquad u_{i,1} = u_{i,-1}.$$

Then from (4) with $j = 0$ we find that

$$u_{i,1} = 2u_{i,0} - u_{i,-1} + \lambda^2(u_{i+1,0} - 2u_{i,0} + u_{i-1,0}) \qquad \Rightarrow \qquad u_{i,1} = (1-\lambda^2)u_{i,0} + \frac{\lambda^2}{2}(u_{i+1,0} - u_{i-1,0}).$$