MA4006: Engineering Mathematics 5 (Spring 2008) Solutions (Sarah Mitchell)

Question 1

a) $\underline{r}'(t) = \begin{cases} (2t, 3t^2, 0) & -1 \le t \le 1 \\ (2, 0, 1) & 1 \le t \le 2 \end{cases}$ & $|\underline{r}'(t)| = \begin{cases} \sqrt{4t^2+9t^4} & -1 \le t \le 1 \\ \sqrt{5} & 1 \le t \le 2 \end{cases}$

$$\therefore \hat{\underline{t}} = \frac{\underline{r}'(t)}{|\underline{r}'(t)|} = \begin{cases} \dfrac{(2t, 3t^2, 0)}{\sqrt{4t^2+9t^4}} & -1 \le t \le 1 \\ \dfrac{(2,0,1)}{\sqrt{5}} & 1 \le t \le 2 \end{cases}$$

It is not smooth since $\hat{\underline{t}}$ is not continuous at $t=1$.
But it is piecewise smooth.

b) $\underline{r}'(t) = 2\cos 2t\,\underline{i} + 4\sin t\cos t\,\underline{j} = 2\cos 2t\,\underline{i} + 2\sin 2t\,\underline{j}$

$\therefore |\underline{r}'(t)| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2$

So arclength $s = \int_0^t |\underline{r}'(\hat{t})|\,d\hat{t} = 2t$. $\Rightarrow t = s/2$

Intrinsic Eqn is then $\underline{r}(s) = \sin s\,\underline{i} + 2\sin^2 \frac{s}{2}\,\underline{j}$.

$\Rightarrow \underline{r}'(s) = \cos s\,\underline{i} + 2\sin\frac{s}{2}\cos\frac{s}{2}\,\underline{j} = \cos s\,\underline{i} + \sin s\,\underline{j}$

$\therefore \underline{r}''(s) = -\sin s\,\underline{i} + \cos s\,\underline{j}$

$\therefore$ curvature $\kappa = |\underline{r}''(s)| = \sqrt{\sin^2 s + \cos^2 s} = 1$.

c) $D_{\underline{b}} f = \nabla f \cdot \hat{\underline{b}}$ $\qquad \hat{\underline{b}} = \dfrac{(1,1,-1)}{\sqrt{3}}$

$\nabla f = \left(\frac{2}{x},\ z\cos(yz),\ y\cos(yz)\right)$

$\therefore D_{\underline{b}} f = \frac{1}{\sqrt{3}}\left[\frac{2}{x} + z\cos(yz) - y\cos(yz)\right]$

Hence the directional derivative at pt $P(1,1,\pi)$ is

$$D_{\underline{b}} f = \frac{1}{\sqrt{3}}\left[2 + \pi\cos\pi - \cos\pi\right]$$
$$= \frac{1}{\sqrt{3}}\left[3 - \pi\right].$$

Question 2

(a) $\underline{r}(t) = (3\cos t, 3\sin t, t)$, $\underline{F}(\underline{r}(t)) = (t, 0, 3\sin t)$

$\underline{r}'(t) = (-3\sin t, 3\cos t, 1)$

Now, the pt $(3,0,0)$ corresponds to $t=0$

pt $(0,3,\frac{\pi}{2})$ " " $t=\frac{\pi}{2}$

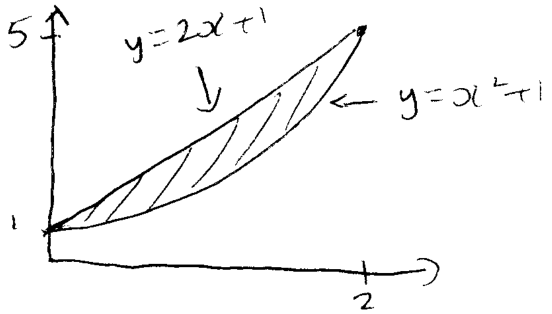
$\therefore$ Work Done $= \int_C \underline{F}\cdot d\underline{r} = \int_0^{\frac{\pi}{2}} \underline{F}(\underline{r}(t))\cdot\underline{r}'(t)\,dt$

$= \int_0^{\frac{\pi}{2}} (-3t\sin t + 3\sin t)\,dt$

$= -3\left[\sin t - t\cos t\right]_0^{\frac{\pi}{2}} + 3\left[-\cos t\right]_0^{\frac{\pi}{2}}$

$= 0$

(b)



So $\iint_R x\,dA = \int_0^2 \int_{x^2+1}^{2x+1} x\,dy\,dx$

$= \int_0^2 \left[xy\right]_{x^2+1}^{2x+1} dx$

$= \int_0^2 \left[2x^2 - x^3\right] dx$

$= \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2$

$= 16/3 - 4$

$= 4/3$

(c) Stokes' Thm: $\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\nabla\times\underline{F})\cdot d\underline{S}$

where S is an oriented smooth surface bounded by a simple, close, smooth boundary curve C with positive orientation

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+y-2z & 2x-4y+z^2 & x+2y-z^2 \end{vmatrix} = \underline{i}(2-2z) - \underline{j}(1+2) + \underline{k}(2-1)$$

$= (2-2z)\underline{i} - 3\underline{j} + \underline{k}$

Let C be the circle that bounds the disk S where $z=3$, $x^2+y^2 \le 25$. Note that unit normal $\underline{n} = \underline{k}$ on S

So by Stokes' Thm:

$\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\nabla\times\underline{F})\cdot d\underline{S} = \iint_S (\nabla\times\underline{F})\cdot\underline{n}\,dS = \iint_S dS = \text{Area of } S = 25\pi$

For parametrize S by $\underline{r} = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j} + 3\underline{k}$ $\quad 0\le\theta\le 2\pi$, $0\le r\le 5$.

Then $\underline{r}_r \times \underline{r}_\theta = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = r\underline{k}$

$(\nabla\times\underline{F})(\underline{r}(r,\theta)) = (-4,-3,1) \Rightarrow \oint_C \underline{F}\cdot d\underline{r} = \int_0^{2\pi}\int_0^5 r\,dr\,d\theta = 25\pi$

Question 3

(a) $\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax^3y^2z & z^3+Bx^4yz & 3yz^2-x^4y^2 \end{vmatrix}$

$= \underline{i}[(\cancel{3z^2} - 2x^4y) - (\cancel{3z^2} + Bx^4y)]$
$- \underline{j}[-4x^3y^2 - Ax^3y^2]$
$+ \underline{k}[4Bx^3yz - 2Ax^3yz]$

choose $A = -4$ & $B = -2$. Then $\nabla \times \underline{F} = \underline{0}$

is $\underline{F} = \nabla \phi$ then

$-4x^3y^2z = \frac{\partial \phi}{\partial x}$ (i)

$z^3 - 2x^4yz = \frac{\partial \phi}{\partial y}$ (ii)

$3yz^2 - x^4y^2 = \frac{\partial \phi}{\partial z}$ (iii)

Integrate (i) w.r.t. x:

$\phi = -x^4y^2z + g(y,z)$

$\therefore \frac{\partial \phi}{\partial y} = -2x^4yz + \frac{\partial g}{\partial y}$

compare with (ii): $\frac{\partial g}{\partial y} = z^3$

$\Rightarrow g = z^3y + h(z)$

So $\phi = -x^4y^2z + z^3y + h(z)$

$\Rightarrow \frac{\partial \phi}{\partial z} = -x^4y^2 + 3yz^2 + h'(z)$

So, comparing with (iii) gives $h'(z) = 0$
$\Rightarrow h(z) = c$.

$\therefore \phi = -x^4y^2z + z^3y + c$.

b) level surfaces of $w(x,y,z)$ are those surfaces defined by $w(x,y,z) = c$, c is an arbitrary const.

Suppose curve C lies on surface S, which is described parametrically by $\underline{r}(t)$. Then $w[\underline{r}(t)] = c$

$\Rightarrow \nabla w \cdot \underline{r}'(t) = 0$

So ∇w is perpendicular to the tangent plane of the surface.

Question 3 cont

consider $Z = e^{xy}$, write as $w(x,y,z) = 0$

ie $w(x,y,z) = z - e^{xy} = 0$

So $\nabla w = (-ye^{xy}, -xe^{xy}, 1)$

At pt $P(1,0,1)$, $\nabla w = (0,-1,1)$

$\Rightarrow |\nabla w| = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore$ unit normal vector $= \frac{1}{\sqrt{2}}(0,-1,1)$

(c) $f(x,y) = f(x_0,y_0) + \delta\underline{r} \cdot \nabla f|_{(x_0,y_0)} + O(|\delta \underline{r}|^2)$

$(x_0,y_0) = (1,0)$ & so $\delta\underline{r} = (x-x_0, y-y_0) = (x-1, y)$

$\nabla f = (2x, e^y)$

$\therefore \nabla f|_{(x_0,y_0)} = (2,1)$ & $f(x_0,y_0) = 2$

So $f(x,y) = 2 + (x-1,y).(2,1) + O(|\delta\underline{r}|^2)$

$= 2 + 2(x-1) + y + O(|\delta\underline{r}|^2)$

$= 2x + y + O(|\delta\underline{r}|^2)$.

So $f(0{\cdot}8, 0{\cdot}1) = 2(0{\cdot}8) + 0{\cdot}1 + O((-0{\cdot}2)^2 + 0{\cdot}1^2)$

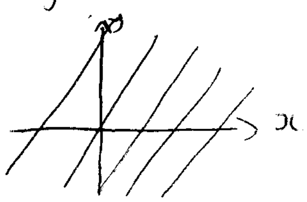
$= 1{\cdot}7 + O(0{\cdot}05)$.

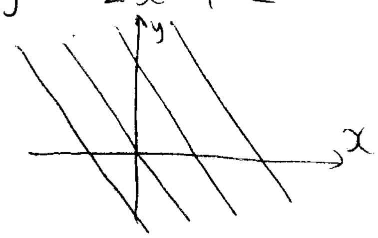
Question 4

(a) A parabolic eqn satisfies $B^2 - AC = 0$
An elliptic eqn satisfies $B^2 - AC < 0$
A hyperbolic eqn satisfies $B^2 - AC > 0$.

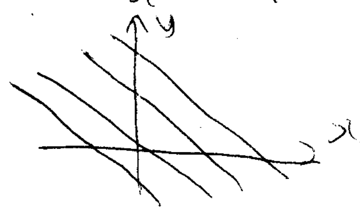
(b) Differential eqn satisfies
$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

(c) (i) $A = 1$, $B = 0$, $C = -4$
$\therefore B^2 - AC = 0 - (1)(-4) = 4 > 0$
Hence PDE is <u>hyperbolic</u>.
Characteristics are $\frac{dy}{dx} = \frac{\pm\sqrt{4}}{1} = \pm 2$
$\therefore y = 2x + c$ or $y = -2x + c$





(ii) $A = 1$, $B = -1$, $C = 1$
$\therefore B^2 - AC = (-1)^2 - (1)(1) = 0 \Rightarrow$ PDE is <u>parabolic</u>
Characteristics are $\frac{dy}{dx} = \frac{-1}{1} = -1$ (only one).
$\therefore y = -x + c$



(iii) $A = 1$, $B = 0$, $C = 3$
$\therefore B^2 - AC = 0 - (1)(3) = -3 < 0 \Rightarrow$ PDE is <u>elliptic</u>
$\Rightarrow$ no characteristics

(d) $\alpha = x - t$, $\beta = x - \frac{1}{2}t$

$u_x = u_\alpha + u_\beta \quad \therefore u_{xx} = u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta}$

$u_t = -u_\alpha - \frac{1}{2}u_\beta \quad \therefore u_{tt} = u_{\alpha\alpha} + u_{\alpha\beta} + \frac{1}{4}u_{\beta\beta}$

Also $u_{xt} = -u_{\alpha\alpha} - \frac{3}{2}u_{\alpha\beta} - \frac{1}{2}u_{\beta\beta}$

$\therefore u_{xx} + 3u_{xt} + 2u_{tt} = -\frac{1}{2}u_{\alpha\beta}$

$\Rightarrow u_{\alpha\beta} = 0$

Question 4 cont.

$$\therefore u(\alpha, \beta) = f(\alpha) + g(\beta)$$

$$\Rightarrow u(x,t) = f(x-t) + g(x - \tfrac{1}{2}t)$$

$$\Rightarrow u_t(x,t) = -f'(x-t) - \tfrac{1}{2}g'(x - \tfrac{1}{2}t)$$

b.c.s $u(x,0) = x^2 \Rightarrow f(x) + g(x) = x^2 \quad (1)$

$u_t(x,0) = 0 \Rightarrow -f'(x) - \tfrac{1}{2}g'(x) = 0$

$\therefore f(x) + \tfrac{1}{2}g(x) = c$

(1) − (2) $\Rightarrow \tfrac{1}{2}g(x) = x^2 + c$

$\Rightarrow g(x) = 2x^2 + 2c$

$\therefore f(x) = x^2 - g(x) = -x^2 - 2c$

$\therefore u(x,t) = -(x-t)^2 + 2(x - \tfrac{1}{2}t)^2.$

Question 5

a) Set $u(x,t) = F(x)G(t)$.

PDE is $F''(x)G(t) = \frac{1}{c^2}F(x)\dot{G}(t)$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{\dot{G}(t)}{c^2 G(t)} = k \quad \text{where } k \text{ is const.}$$

Two ODEs: $F''(x) - kF(x) = 0$

$\dot{G}(t) - kc^2 G(t) = 0$

BCs $u(0,t) = 0 \Rightarrow F(0) = 0$

$u_x(L,t) = 0 \Rightarrow F'(L) = 0$

Solve $F''(x) - kF(x) = 0$

$F(0) = F'(L) = 0$.

- set $k = 0$: $F''(x) = 0 \Rightarrow F(x) = Ax + B$

  BCs $\Rightarrow A = B = 0$. $\therefore F(x) = 0$.

- set $k = p^2 > 0$ $\quad F(x) = Ae^{px} + Be^{-px}$

  BCs $\Rightarrow A + B = 0$ & $p(Ae^{pL} - Be^{-pL}) = 0$

  $\therefore A = B = 0 \Rightarrow F(x) = 0$.

- set $k = -p^2 < 0$ $\quad F(x) = A\cos px + B\sin px$

  BC $F(0) = 0 \Rightarrow A = 0$.

  $F'(x) = Bp\cos px$

  $\therefore F'(L) = 0 \Rightarrow \cos pL = 0 \Rightarrow pL = (n - \tfrac{1}{2})\pi$, $n = 1, 2, \ldots$

  $\therefore k = -p^2 = -\frac{(n-\frac{1}{2})^2\pi^2}{L^2}$. & $F_n(x) = B_n \sin\frac{(n-\frac{1}{2})\pi x}{L}$.

put $\lambda_n^2 = -p^2c^2 = -\frac{c^2(n-\frac{1}{2})^2\pi^2}{L^2}$

& then solve $\dot{G}_n(t) + \lambda_n^2 G_n(t) = 0$

$\Rightarrow G_n(t) = D_n e^{-\lambda_n^2 t}$.

So $u_n(x,t) = F_n(x)G_n(t)$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{c^2(n-\frac{1}{2})^2\pi^2 t}{L^2}\right)\sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$$

IC $u(x,0) = \sin\frac{3\pi x}{2L}$

ie. $\sin\frac{3\pi x}{2L} = \sum_{n=1}^{\infty} C_n \sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$

by inspection, $C_n = 0$ for all n except $n = 2$ where $C_2 = 1$.

Question 5 cont.

Hence $u(x,t) = \exp\left(\frac{-9c^2\pi^2 t}{4l^2}\right)\sin\left(\frac{3\pi x}{2l}\right)$.

(b) $u = e^{-kt}\cos mx \cos nt$

$u_t = -ke^{-kt}\cos mx\cos nt - ne^{-kt}\cos mx \sin nt$

$= e^{-kt}\cos mx[-k\cos nt - n\sin nt]$

$u_{tt} = -ke^{-kt}\cos mx[-k\cos nt - n\sin nt]$

$+ e^{-kt}\cos mx[kn\sin nt - n^2\cos nt]$

$= e^{-kt}\cos mx[k^2\cos nt + 2kn\sin nt - n^2\cos nt]$

Also $u_x = -me^{-kt}\sin mx\cos nt$

$u_{xx} = -m^2e^{-kt}\cos mx\cos nt$

So

$c^2u_{xx} - u_{tt} - 2ku_t$

$= -c^2m^2e^{-kt}\cos mx\cos nt$

$-e^{-kt}\cos mx[k^2\cos nt + 2k\cancel{n\sin nt} - n^2\cos nt]$

$-2ke^{-kt}\cos mx[-k\cos nt - \cancel{n\sin nt}]$

$= e^{-kt}\cos mx\cos nt[-c^2m^2 + k^2 + n^2]$

$= 0 \implies \boxed{c^2m^2 = k^2 + n^2}$

Question 6

(a) The bvp $u_{xx} + u_{yy} = f(x,y)$ is discretised as

$$u(x+h,y) + u(x-h,y) + u(x,y+h) - u(x,y-h) - 4u(x,y) \approx h^2 f(x,y) \quad (*)$$

The square is discretised by a mesh (uniform) of width $h = 1/N$.
The mesh pts are the intersection of the lines (x_i, y_j) where

$x_i = ih$ & $y_j = jh$ for some $0 \leq i, j \leq N$.

We write the approx to $u(x_i, y_j)$ as $U_{i,j}$. Then $(*)$ becomes

$$U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j} = h^2 f_{i,j}$$

where $f_{i,j} = f(x_i, y_j)$

on the boundary:

$U_{i,0} = \phi_0(x_i)$, $U_{i,N} = \phi_1(x_i)$

$U_{0,j} = \psi_0(y_j)$, $U_{N,j} = \psi_1(y_j)$

(b) Assume for contradiction that there is some i_0 & j_0 s.t.

$$V_{i_0,j_0} = \max_{\Omega^h} V_{i,j}$$

& $$V_{i_0,j_0} > \max_{\Gamma^h} V_{i,j}$$

Then $L^h V_{i_0,j_0} \geq 0 \Rightarrow V_{i_0,j_0} \leq \frac{1}{4}[V_{i_0+1,j_0} + V_{i_0-1,j_0} + V_{i_0,j_0+1} + V_{i_0,j_0-1}]$

But $V_{i_0,j_0} \geq \max[V_{i_0+1,j_0}, V_{i_0-1,j_0}, V_{i_0,j_0+1}, V_{i_0,j_0-1}]$

$\Rightarrow V_{i_0,j_0} = V_{i_0+1,j_0} = V_{i_0-1,j_0} = V_{i_0,j_0+1} = V_{i_0,j_0-1}$

if any of these are on the boundary we get a contradiction, else repeat argument by applying L^h to each of the mesh functions & so on until a contradiction is reached.

Question 6 cont

(c) If we let $u_{i,j}^{(1)}$ & $u_{i,j}^{(2)}$ be two solutions of

$$L^h u_{i,j} = f(x_i, y_j)$$

with
$$u_{i,0} = \phi_0(x_i), \quad u_{i,N} = \phi_1(x_i)$$
$$u_{0,j} = \psi_0(y_j), \quad u_{N,j} = \psi_1(y_j)$$

then
$$L^h(u_{i,j}^{(1)} - u_{i,j}^{(2)}) = f(x_i, y_j) - f(x_i, y_j) = 0$$

(an application of the maximum principle).

& also $u_{i,j}^{(1)} - u_{i,j}^{(2)} = 0$ on the boundaries

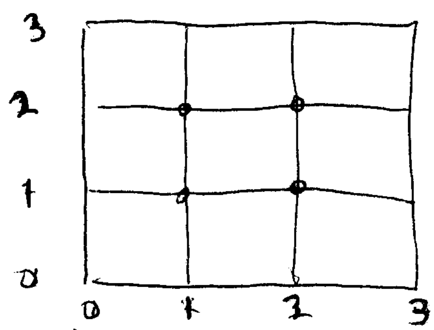
so that $u_{i,j}^{(1)} - u_{i,j}^{(2)} = 0$ on $\bar{\Omega}^h$

ie. the solution is unique $(u_{i,j}^{(1)} = u_{i,j}^{(2)})$

(d) Using $h = \frac{1}{3}$ means there are just 4 internal pts.

Difference eqn to be solved is therefore

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2(x_i y_j + 1) \qquad i = 1, 2, \ j = 1, 2$$



Given $u_{i,0} = x_i(1 - x_i), \quad u_{i,N} = 1 \qquad i = 0, 1, 2, 3,$

$u_{0,j} = \frac{y_j}{2}(1 + y_j), \quad u_{N,j} = y_j \qquad j = 0, 1, 2, 3,$

Hence system of eqns to solve is

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = h^2(x_1 y_1 + 1)$$
$$u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} - 4u_{2,1} = h^2(x_2 y_1 + 1)$$
$$u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} - 4u_{1,2} = h^2(x_1 y_2 + 1)$$
$$u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1} - 4u_{2,2} = h^2(x_2 y_2 + 1)$$

To determine the 4 unknowns $u_{1,1}$, $u_{1,2}$, $u_{2,1}$ & $u_{2,2}$