UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics and Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4006

SEMESTER: Spring 2011

MODULE TITLE: Engineering Mathematics 5

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. Sarah Mitchell

PERCENTAGE OF TOTAL MARKS: 80%

EXTERNAL EXAMINER: Prof. Tim Myers

INSTRUCTIONS TO CANDIDATES:
Answer any **two** questions from Section A and any **two** questions from Section B.
There are some **useful formulae** on the back page of this booklet.
You may use a calculator and log tables.

SECTION A

Answer any two questions

1. (a) Show that the vector field $\mathbf{F} = yz^2\mathbf{i} + (xz^2 + 2)\mathbf{j} + (2xyz - 1)\mathbf{k}$ is conservative. Determine an associated *scalar potential* ϕ for this vector field $\mathbf{F}$. Hence find $\int_C \mathbf{F} \cdot \mathbf{dr}$ where C is any curve starting at the point $(1, 1, 0)$ and ending at the point $(1, 2, 1)$. 11

   (b) Explain the term *level surfaces* in relation to any scalar field $\Omega(x, y, z)$. Interpret $\nabla\Omega$ with regard to level surfaces. Hence find the unit normal to the surface $z = x^2 \cos y$, at the point $(1, \pi/2, 0)$. 8

   (c) Find the directional derivative of the scalar valued function

   $$f(x, y, z) = -\ln(x^2 + y^2 + z^2)$$

   at the point $(-1,\ 0,\ 1)$ in the direction of the vector $2\mathbf{i} - \mathbf{k}$. 6

2. (a) Evaluate $\int_C xy^4\, ds$ where C is the right half of the circle $x^2 + y^2 = 16$, $x \geq 0$, oriented in the anticlockwise direction. 7

   (b) Calculate the surface area of the cone S defined by $\mathbf{r}(r, \theta) = (r\cos\theta,\ r\sin\theta,\ r)$ for $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$. 6

   (c) Suppose S is the surface of the closed region $x^2 + y^2 \leq 4$, $-3 \leq z \leq 3$.

   (i) Sketch the surface. 2

   (ii) State the divergence theorem. 2

   (iii) Use the divergence theorem to evaluate $\iint_S \mathbf{f} \cdot \mathbf{dS}$ where $\mathbf{f} = \big(x^3, y^3, \cos(x^3y^3)\big)$. 8

3. (a) Consider the vector valued function

   $$\mathbf{r}(t) = \begin{cases} (t^2,\ 4\cos t,\ e^{2t}) & \text{if } -1 \leq t \leq 0 \\ (2t,\ 4 - t,\ 2 - e^{-t}) & \text{if } 0 \leq t \leq 2. \end{cases}$$

   Find its derivative, and hence determine the **unit** tangent vector to the curve defined by $\mathbf{r}(t)$ and comment on its smoothness. 9

   (b) Show that the helix $\mathbf{r}(t) = (a\cos t,\ a\sin t,\ ct)$ can be represented by $(a\cos(s/k),\ a\sin(s/k),\ cs/k)$ where $k = \sqrt{a^2 + c^2}$ and s is the arclength. Find an expression for the curvature. 9

(c) Use Taylor's theorem in three dimensions to find a first order approximation for

$$f(x,y,z) = xy^2 + yz$$

at the point $(1,1,2)$. Hence estimate $f(0.9,1.2,2.3)$. 7

SECTION B

Answer any two questions

4. Consider the second order partial differential equation

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = f(x,y,u_x,u_y),$$

where A, B, C and f are all known functions. This equation has characteristics defined by the differential equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

(a) Give the criteria to classify the above equation as hyperbolic, elliptic or parabolic. 3

(b) Consider the following partial differential equations:

(i) $u_t = 4u_{xx} + u_x$

(ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} + 3\dfrac{\partial^2 u}{\partial y^2} = u$

(iii) $u_{rr} - 6u_{r\theta} + u_{\theta\theta} = 0.$

Fully classify these equations and find and sketch the real characteristics (if any). 9

(c) Consider the partial differential equation

$$6u_{xx} + 5u_{xy} - u_{yy} = 0.$$

Using the transformation $\xi = x + 6y$, $\eta = x - y$, reduce the equation to a simpler form and determine the particular solution that satisfies the boundary conditions

$$u(x,0) = \sin(2x) \qquad \text{and} \qquad u_y(x,0) = \cos(2x).$$

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5. (a) Consider the Laplace equation $u_{xx} + u_{yy} = 0$ describing steady heat conduction in the region $0 \leq x \leq l$, $0 \leq y < \infty$, which is zero on the boundaries $x = 0$, $x = l$ and as $y \to \infty$, and takes a constant value u_0 on $y = 0$, $0 < x < l$. Use the method of separation of variables to show that the solution is given by

$$u(x,y) = \frac{4u_0}{\pi} \sum_{\text{odd } n} \frac{1}{n} e^{-n\pi y/l} \sin \frac{n\pi x}{l}.$$

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(b) Use Laplace transforms to solve the heat equation $u_t = u_{xx}$ on the semi-infinite domain $0 \leq x < \infty$, $0 \leq t < \infty$, subject to $u(x,0) = 0$, $u(0,t) = t$ and $u \to 0$ as $x \to \infty$. 8

(c) Show that the equation

$$\theta_t = c^2\theta_{xx} - \alpha(\theta - \theta_0)$$

can be reduced to the standard heat equation $u_t = c^2 u_{xx}$ by writing $u = e^{\alpha t}(\theta - \theta_0)$. 4

6. (a) Consider the finite difference approximation

$$f''(x) \cong \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Use Taylor's series to show that this has $O(h^2)$ accuracy. 4

(b) Let Ω be the unit square $(0,1) \times (0,1)$, with boundary Γ and closure $\bar{\Omega} = [0,1] \times [0,1]$. Consider the boundary value problem

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x,y), \quad (x,y) \in \Omega,$$

with boundary conditions

$$\begin{aligned} u(x,0) &= \phi_0(x), & u(x,1) &= \phi_1(x) \\ u(0,y) &= \psi_0(y), & u(1,y) &= \psi_1(y). \end{aligned}$$

Taking a fixed-width mesh h in both the x and y directions, formulate the discretised problem and outline a uniform mesh. The second derivative can be approximated using the standard $O(h^2)$ operator from (a). 6

(c) Use the result from part (b) to determine the linear system of equations resulting from discretising the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1$$

describing steady heat conduction with a heating term, in the unit square, given that

$$u(0, y) = 0, \qquad u(1, y) = y^2,$$
$$u(x, 0) = 0, \qquad u(x, 1) = x$$

Use a stepsize of $h = \frac{1}{3}$ in both the x and y directions. 8

(d) Consider the wave equation $u_{tt} = c^2 u_{xx}$ on $0 \leq x \leq 1$ and $t > 0$, with

$$u(x, 0) = x(1 - x), \quad \frac{\partial u}{\partial t}(x, 0) = 0, \quad u(0, t) = 0, \quad u(1, t) = 0.$$

Formulate the discretised problem using an explicit finite difference method on a uniform mesh (taking central differences to approximate the second derivative). 7

Useful Information

- Cylindrical co-ordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

  where the Jacobian determinant is given by $J = r$.

- Taylor Series in more than one variable:

$$f(x,y,z) = f(x_0,y_0,z_0) + \delta\mathbf{r}\cdot\nabla f|_{(x_0,y_0,z_0)} + O(|\delta\mathbf{r}|^2)$$

  where $\delta\mathbf{r} = (h,k,l)$ with $h = x - x_0$, $k = y - y_0$ and $l = z - z_0$.

- Half-range Fourier **sine** series of period $2l$:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$\text{with} \quad B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

- Useful Laplace transform result. If

$$\hat{f}(s) = \frac{1}{s^2}e^{-k\sqrt{s}},$$

  then

$$f(t) = \left(t + \frac{x^2}{2}\right)\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \sqrt{\frac{t}{\pi}}\,x e^{-x^2/[4t]}.$$