UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics and Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4006

SEMESTER: Spring 2010

MODULE TITLE: Engineering Mathematics 5

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. Sarah Mitchell

PERCENTAGE OF TOTAL MARKS: 80%

EXTERNAL EXAMINER: Prof. Jim Flavin

INSTRUCTIONS TO CANDIDATES:
Answer two questions from Section A and two questions from Section B.
There are some useful formulae on the back page of this booklet.
You may use a calculator and log tables.

SECTION A

Answer any two questions

1. (a) Consider the vector valued function

$$\mathbf{r}(t) = \begin{cases} (t,\, 2,\, \ln t) & \text{if } 0.5 \le t \le 1 \\ (1,\, 4-2t,\, t-1) & \text{if } 1 \le t \le 2. \end{cases}$$

Find its derivative, and hence find the unit tangent vector to the curve defined by $\mathbf{r}(t)$ and comment on its smoothness 10

(b) Find an expression for the arclength of the curve expressed parametrically as

$$\mathbf{r}(t) = (3\cos t,\, 3\sin t,\, 2t)\ , \qquad 0 \le t \le \pi.$$

Using the arclength find the intrinsic equation of the curve represented by $\mathbf{r}(t)$. Find an expression for the curvature of this curve at any point. 8

(c) Find the directional derivative of the scalar valued function

$$f(x, y, z) = x^2yz + xy^2z + xyz^2$$

at the point (1, 1, 0) in the direction of the vector $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. 7

2. (a) Express $\operatorname{curl}(\operatorname{grad}\omega)$ and $\operatorname{div}(\operatorname{curl}\mathbf{F})$ in operator notation and hence evaluate these when $\omega = xy/z$ and $\mathbf{F} = (\sin x, \cos y, xy)$. 6

(b) Show that $\mathbf{f} = z^2\mathbf{j} + 2yz\mathbf{k}$ is a conservative vector field. Determine an associated *scalar potential* ϕ for this vector field $\mathbf{f}$. 8

(c) Using the fact that $\nabla\Omega$ (where $\Omega = \Omega(x, y)$ is a scalar valued function of position) is perpendicular to its own level curves, find a unit normal to the curve defined by $y - x^2 = 0$ at the point $(1, 1)$. Verify this result with a rough sketch. 5

(d) Use Taylor's theorem in two dimensions to find a first order approximation for

$$f(x, y) = \sin(xy) + x^2 + y,$$

at the point $(0, 1)$. Hence estimate $f(0.1, 1.2)$. 6

3. (a) Find the work done in moving a particle in a force field given by

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k},$$

along the curve C with parametric definition, $\mathbf{r}(t) = (\cos t, \sin t, t)$, from the point $(1, 0, 0)$ to the point $(1, 0, 4\pi)$ and sketch the curve C. 8

(b) Evaluate $\iint_S (x + z)\, dS$ where S is part of the plane $x + y + z = 1$ that lies in the first octant. 8

(c) State Green's Theorem in the plane.
Use Green's theorem to find $\oint_C y^3\, dx - x^3\, dy$ where C is the positively oriented circle of radius 2 centred at the origin. 9

SECTION B

Answer any two questions

4. Consider the second order partial differential equation

$$A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} = f(x, y, u_x, u_y),$$

where A, B, C and f are all known functions. This equation has characteristics defined by the differential equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

(a) Give the criteria to classify the above equation as hyperbolic, elliptic or parabolic. 3

(b) Consider the following partial differential equations:

(i) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$

(ii) $4\dfrac{\partial^2 u}{\partial x^2} + 12\dfrac{\partial^2 u}{\partial x \partial t} + 9\dfrac{\partial^2 u}{\partial t^2} - 2\dfrac{\partial u}{\partial x} + u = 0$

(iii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

Fully classify these equations and find and sketch the real characteristics (if any). 9

(c) Consider the partial differential equation

$$4u_{xx} + 5u_{xy} + u_{yy} = 0.$$

Using the transformation $\alpha = x - 4y$, $\beta = x - y$, reduce the equation to a simpler form and determine the particular solution that satisfies the boundary conditions

$$u(x, 0) = \cos x \qquad \text{and} \qquad u_y(x, 0) = 0.$$

13

5. (a) The longitudinal oscillations of air in a pipe of length l are described by the linear wave equation $u_{tt} = c^2 u_{xx}$ where $u = u(x, t)$. Assume that the air in the pipe is initially at rest and that the initial displacement is given by $u(x, 0) = f(x)$. Furthermore, assume that the two ends of the pipe are closed (*i.e.*, $u = 0$ at each end).

   Set up the appropriate initial boundary value problem modelling the subsequent oscillations of the air. Use the method of separation of variables to solve this problem. 13

   (b) Use Laplace transforms to solve the wave equation $u_{tt} = c^2 u_{xx}$, $c = 1$, on the semi-infinite domain $0 \le x < \infty$, $0 \le t < \infty$, subject to $u(x, 0) = 0$, $u_t(x, 0) = 0$, $u(0, t) = e^{-t}$ and $u \to 0$ as $x \to \infty$. 8

   (c) Verify that $u(x, t) = \sin 4t \cos x$ is a solution of the wave equation $u_{tt} = c^2 u_{xx}$ for a suitable value of the constant c. Identify this value of c. 4

6. (a) Consider the finite difference approximation

$$f'(x) \cong \frac{-f(x + 2h) + 4f(x + h) - 3f(x)}{2h}.$$

   Show that this has $O(h^2)$ accuracy. 5

   (b) Let Ω be the unit square $(0, 1) \times (0, 1)$, with boundary Γ and closure $\bar{\Omega} = [0, 1] \times [0, 1]$. Consider the boundary value problem

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega,$$

with boundary conditions

$$u(x,0) = \phi_0(x), \qquad u(x,1) = \phi_1(x)$$
$$u(0,y) = \psi_0(y), \qquad u(1,y) = \psi_1(y).$$

Taking a fixed-width mesh h in both the x and y directions, formulate the discretised problem and outline a uniform mesh. The second derivative can be approximated using the standard $O(h^2)$ operator. 5

(c) Using the result from part (b), determine the linear system of equations resulting from the discretised problem, with $f(x,y) = f = \sin(x+y)$, and boundary conditions given by

$$u(x,0) = 1, \qquad u(x,1) = x(1-x),$$
$$u(0,y) = 1-y, \qquad u(1,y) = 1-y^2.$$

Use a stepsize of $h = \frac{1}{3}$ in both the x and y directions. 8

(d) Consider the heat equation $u_t = c^2 u_{xx}$ for a bar of conducting material of length 1, subject to

$$u(x,0) = x^2, \qquad \frac{\partial u}{\partial x}(0,t) = 0, \qquad u(1,t) = 1.$$

Formulate the discretised problem using an explicit finite difference method on a uniform mesh (taking central differences to approximate the second derivative). 7

Useful Information

- Polar co-ordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

  where the Jacobian determinant is given by $J = r$.

- Taylor Series in more than one variable:

$$f(x,y,z) = f(x_0,y_0,z_0) + \delta\mathbf{r}\cdot\nabla f|_{(x_0,y_0,z_0)} + O(|\delta\mathbf{r}|^2)$$

  where $\delta\mathbf{r} = (h,k,l)$ with $h = x - x_0$, $k = y - y_0$ and $l = z - z_0$.

- Half-range Fourier **sine** series of period $2l$:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$\text{with} \quad B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

- Useful Laplace transform results. Suppose $\mathbf{L}(f) = \hat{f}(s) = \int_0^\infty f(t)e^{-st}dt$. Then the t *shift* property states that

$$\mathbf{L}^{-1}\left[e^{-as}\hat{f}(s)\right] = H(t-a)f(t-a).$$

  Also

$$\hat{f}(s) = \frac{1}{1+s} \qquad \Longrightarrow \qquad f(t) = e^{-t}.$$