UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics and Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4006

SEMESTER: Spring 2009

MODULE TITLE: Engineering Mathematics 5

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. Sarah Mitchell

PERCENTAGE OF TOTAL MARKS: 70%

INSTRUCTIONS TO CANDIDATES:
Answer two questions from Section A and two questions from Section B.
There are some useful formulae on the back page of this booklet.
You may use a calculator and log tables.

SECTION A

Answer any two questions

1. (a) Show that the first derivative of a unit vector $\hat{\mathbf{u}} = \hat{\mathbf{u}}(t)$ is always perpendicular to $\hat{\mathbf{u}}$ provided the derivative is not zero. 3

   (b) A particle moves along a curve whose parametric equations are

$$x = 2\cos 3t, \qquad y = 2\sin 3t, \qquad z = 4t,$$

   at time t.

   (i) Sketch the curve. 2

   (ii) Determine its velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ for a general t. Hence find the magnitudes of $\mathbf{v}$ and $\mathbf{a}$ at $t = 0$. 4

   (iii) Show that $\mathbf{v}$ is perpendicular to $\mathbf{a}$ and explain why this could **not** have been deduced directly from the result in part (a), even though $\mathbf{a} = \mathbf{v}'$. 3

   (iv) Find an expression for the arclength from $t = 0$ to an arbitrary point on the curve. 4

   (v) Write down the equation for the curve in intrinsic form and find an expression for the curvature. 3

   (c) Find the directional derivative of the scalar valued function

$$f(x, y, z) = 2x^2 + 3y^2 + z^2,$$

   at the point $(2, 1, 3)$ in the direction of the vector $\mathbf{i} - 2\,\mathbf{j}$. 6

2. (a) Show that the vector field

$$\mathbf{F} = (y^2z^3\cos x - 4x^3z)\,\mathbf{i} + 2z^3y\sin x\,\mathbf{j} + (3y^2z^2\sin x - x^4)\,\mathbf{k},$$

   is conservative. Determine an associated scalar potential for this field. Hence find $\int_C \mathbf{F}\cdot\mathbf{dr}$ where C is any curve starting at the point $(0, 1, 0)$ and ending at the point $(\pi, 1, 1)$. 11

   (b) Explain the term *level surfaces* in relation to any scalar field $\Omega(x, y, z)$. Interpret $\nabla\Omega$ with regard to level surfaces. Hence find the unit normal to the surface $x^2 + 2y^2 + z^2 = 7$, at the point $(1, 1, 2)$. 8

   (c) Use Taylor's theorem in two dimensions to find a first order approximation for

$$f(x, y) = \cos(xy) + e^{xy},$$

   at the point $(2, 0)$. Hence estimate $f(1.5, 0.3)$. 6

3. (a) Find the work done in moving a particle in a force field given by

$$\mathbf{F}(x,y) = 3xy\,\mathbf{i} - y^2\,\mathbf{j},$$

along the curve C in the xy-plane, $y = 2x^2$, from the point $(0,0)$ to the point $(1,2)$. 7

(b) Evaluate the integral $\iint_R e^{x^2+y^2}\,dx\,dy$ where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.
[**Hint**: Transform the integral using polar coordinates]. 8

(c) State the divergence theorem and use it to evaluate the surface integral $\iint_S \mathbf{f}\cdot\mathbf{dS}$ where

$$\mathbf{f} = xy\,\mathbf{i} - \frac{1}{2}y^2\,\mathbf{j} + z\,\mathbf{k},$$

and S consists of the two surfaces $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top and $z = 1$ on the bottom.
[**Hint**: Transform the integral using cylindrical coordinates]. 10

SECTION B

Answer any two questions

4. Consider the second order partial differential equation

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = f(x,y,u_x,u_y),$$

where A, B, C and f are all known functions. This equation has characteristics defined by the differential equation

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

(a) Give the criteria to classify the above equation as hyperbolic, elliptic or parabolic. 3

(b) Consider the following partial differential equations:

(i) $u_{rr} + u_{r\theta} + u_{\theta\theta} + \theta u_r = 0$

(ii) $u_{xx} + 3u_{xy} = e^x u^3$

(iii) $u_{xx} + 2u_{xy} + u_{yy} = u_y.$

Fully classify these equations and find and sketch the real characteristics (if any). 9

(c) Consider the partial differential equation

$$u_{xx} - 2u_{xy} - 8u_{yy} = 0.$$

(i) Determine the constants a and b so that the transformation $\xi = x + ay$, $\eta = x + by$, reduces the equation to the form

$$u_{\xi\eta} = 0.$$

Hence find the solution of the equation in terms of two arbitrary functions. 8

(ii) Determine the particular solution that satisfies the boundary conditions

$$u(x, 0) = \cos(4x), \qquad u_y(x, 0) = -\sin(4x).$$

5

5. (a) The temperature, $u(x, t)$, in a heat conducting bar of length l is governed by the heat equation

$$u_t = c^2 u_{xx},$$

where t is the time and x is the distance along the bar. The ends of the bar are perfectly insulated so that

$$u_x(0, t) = u_x(l, t) = 0.$$

The initial temperature, $u(x, 0)$, in the bar is given by

$$u(x, 0) = \frac{u_0 x}{l}.$$

Use the method of separation of variables to determine the temperature, $u(x, t)$. 16

(b) Use Laplace transforms to solve the heat equation $u_t = c^2 u_{xx}$ on the semi-infinite domain $0 \le x < \infty$, $0 \le t < \infty$, subject to

(i) $u_x(0, t) = -1, \quad t > 0;$

(ii) $u(x, t) \to 0, \quad \text{as } x \to \infty;$

(iii) $u(x, 0) = 0, \quad x > 0.$

9

6. (a) Consider the finite difference approximation

$$f'(x) \cong \frac{2f(x+3h) - 9f(x+2h) + 18f(x+h) - 11f(x)}{6h}.$$

Show that this has $O(h^3)$ accuracy. 5

(b) Let Ω be the unit square $(0,1) \times (0,1)$, with boundary Γ and closure $\bar{\Omega} = [0,1] \times [0,1]$. Consider the boundary value problem

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in \Omega,$$

with boundary conditions

$$u(x,y) = g(x,y), \quad (x,y) \in \Gamma.$$

Taking a fixed-width mesh h in both the x and y directions, formulate the discretised problem and outline a uniform mesh. The 2nd derivative can be approximated using the standard $O(h^2)$ operator. 5

(c) Using the result from part (b), determine the linear system of equations resulting from the discretised problem, with $f(x,y) = 0$, and boundary conditions given by

$$u(x,0) = x(1-x), \qquad u(x,1) = 1,$$
$$u(0,y) = y, \qquad u(1,y) = y.$$

Use a stepsize of $h = \frac{1}{3}$ in both the x and y directions. 8

(d) Consider the wave equation $u_{tt} = c^2 u_{xx}$ for the semi-infinite string $0 \le x < \infty$ and $t \ge 0$, with

$$u(x,0) = xe^{-5(x-1)^2}, \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad u(0,t) = 0.$$

Formulate the discretised problem using an explicit finite difference method on a uniform mesh (taking central differences to approximate the 2nd derivatives). 7

Useful Information

- Cylindrical co-ordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

  where the Jacobian determinant is given by $J = r$.

- Taylor Series in more than one variable:

$$f(x,y,z) = f(x_0,y_0,z_0) + \delta\mathbf{r}\cdot\nabla f|_{(x_0,y_0,z_0)} + O(|\delta\mathbf{r}|^2)$$

  where $\delta\mathbf{r} = (h,k,l)$ with $h = x - x_0$, $k = y - y_0$ and $l = z - z_0$.

- Half-range Fourier **cosine** series of period $2l$:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

$$\text{with} \quad A_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

- Useful inverse Laplace transform: If

$$\hat{f}(s) = \frac{1}{\sqrt{s^3}}e^{-k\sqrt{s}},$$

  then

$$f(t) = 2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right) - k\,\mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right).$$