UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

FACULTY OF SCIENCE AND ENGINEERING
DEPARTMENT OF MATHEMATICS & STATISTICS

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4006 SEMESTER: Spring 2008

MODULE TITLE: Engineering Mathematics 5 DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. Sarah Mitchell PERCENTAGE OF TOTAL MARKS: 70%

INSTRUCTIONS TO CANDIDATES:
The best *four* answered questions contribute to this assessment.
There are some useful formulae on the back page of this booklet.
You may use a calculator and log tables.

1. (a) Consider the vector valued function

$$\mathbf{r}(t) = \begin{cases} (1+t^2,\ t^3,\ 1) & \text{if} \quad -1 \le t \le 1 \\ (2t,\ 1,\ t) & \text{if} \quad 1 \le t \le 2\ . \end{cases}$$

Find its derivative, and hence find the unit tangent vector to the curve defined by $\mathbf{r}(t)$ and comment on its smoothness 10

(b) Consider the path traced out by the point whose position vector at time t is

$$\mathbf{r}(t) = \sin 2t\,\mathbf{i} + 2\sin^2 t\,\mathbf{j}\ , \qquad 0 \le t \le \pi\ .$$

Determine the arclength parameter s and write down the intrinsic equation of the curve. Find an expression for the curvature. 8

(c) Find the directional derivative of the scalar valued function

$$f(x,y,z) = \sin(yz) + \ln(x^2)\ ,$$

at the point $(1,1,\pi)$ in the direction of the vector $\mathbf{i}+\mathbf{j}-\mathbf{k}$. 7

2. (a) Find the work done by the force field

$$\mathbf{F}(x,y,z) = z\,\mathbf{i} + y\,\mathbf{k}\ ,$$

in moving a particle from the point $(3,\ 0,\ 0)$ to the point $(0,\ 3,\ \pi/2)$ along the helix $x = 3\cos t$, $y = 3\sin t$, $z = t$. 7

(b) Evaluate $\iint_R x\,\mathrm{d}A$ where R is the region $x^2+1 \le y \le 2x+1$. 7

(c) State Stokes' Theorem in the plane.

Let C be the circle $x^2+y^2 = 25$ in the plane $z = 3$, oriented counter-clockwise when viewed from above. Use Stokes' Theorem to evaluate $\oint \mathbf{F}\cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = (2x+y-2z)\,\mathbf{i} + \left(2x-4y+z^2\right)\mathbf{j} + \left(x+2y-z^2\right)\mathbf{k}\ .$$

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3. (a) The vector field

$$\mathbf{F}(x,y,z) = Ax^3y^2z\,\mathbf{i} + (z^3 + Bx^4yz)\,\mathbf{j} + (3yz^2 - x^4y^2)\,\mathbf{k}\ ,$$

is conservative. Find values of the constants A and B. Hence find a scalar field ϕ such that $\mathbf{F} = \nabla\phi$. 10

(b) Explain the term *level surfaces* in relation to any scalar field ω. Interpret $\nabla\omega$ with regard to level surfaces. Hence find the unit normal to the surface defined by $z = e^{xy}$ at the point $(1, 0, 1)$. 9

(c) Use Taylor's theorem in two dimensions to find the first order approximation of

$$f(x, y) = e^y + x^2 ,$$

at the point $(1, 0)$. Hence estimate $f(0.8, 0.1)$. 6

4. Consider the second order partial differential equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = f(x, y, u_x, u_y) ,$$

where A, B and C are constant.

(a) Give the criteria to classify the above equation as parabolic, elliptic or hyperbolic. 3

(b) Write down the differential equation which defines the characteristics. 2

(c) Consider the following partial differential equations:

(i) $u_{xx} - 4u_{yy} + u_x = 0$

(ii) $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial u}{\partial x} - u$

(iii) $u_{xx} + 3u_{yy} = \sin x$.

Fully classify these equations and find and sketch the real characteristics (if any). 10

(d) Consider the partial differential equation

$$u_{xx} + 3u_{xt} + 2u_{tt} = 0 .$$

Use the transformation $\alpha = x - t$, $\beta = x - \frac{1}{2}t$ to reduce the equation to a simpler form and determine the general solution $u(x, t)$.

Hence find the particular solution that satisfies the boundary conditions

$$u(x, 0) = x^2 , \qquad u_t(x, 0) = 0 .$$

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5. (a) Consider a bar of heat conducting material of length l. The partial differential equation describing the conduction of heat through the bar is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t} \ .$$

Assume zero temperature at $x = 0$, i.e. $u(0,t) = 0$, and no heat loss at $x = l$, i.e. $u_x(l,t) = 0$.

Use the method separation of variables to show that a form of the solution appropriate to these boundary conditions is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(- \frac{c^2\left(n-\frac{1}{2}\right)^2\pi^2 t}{l^2}\right) \sin\left(\frac{\left(n-\frac{1}{2}\right)\pi x}{l}\right) \ ,$$

where the B_n are constants.

Show that if, in addition, $u = \sin\left(\frac{3\pi x}{2l}\right)$ when $t = 0$ for $0 < x < l$, then the solution simplifies to

$$u(x,t) = \exp\left(-\frac{9c^2\pi^2 t}{4l^2}\right)\sin\left(\frac{3\pi x}{2l}\right) \ .$$

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(b) Find the equation relating the constants k, m, n and c so that the function

$$u = e^{-kt}\cos(mx)\cos(nt) \ ,$$

is a solution of the partial differential equation

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2k\frac{\partial u}{\partial t} \ .$$

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6. Let Ω be the unit square $(0,1)\times(0,1)$, with boundary Γ and closure $\bar{\Omega} = [0,1]\times[0,1]$. Consider the boundary value problem

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y)\in\Omega$$

with Dirichlet boundary conditions

$$u(x,0) = \phi_0(x)\ , \qquad u(x,1) = \phi_1(x)$$
$$u(0,y) = \psi_0(y)\ , \qquad u(1,y) = \psi_1(y)\ .$$

(a) Taking a fixed-width mesh h in both the x and y directions, formulate the discretised problem. The 2nd derivative can be approximated using the standard $O(h^2)$ operator. 4

(b) Show that the finite difference operator L^h satisfies a maximum principle of the form:

$$L^h v_{i,j} \geq 0 \;\forall\, i,j \;\Rightarrow\; \max_{\Omega^h} |v_{i,j}| \leq \max_{\Gamma^h} |v_{i,j}|$$

where $v_{i,j}$ is any mesh-function defined on $\bar{\Omega}^h$. 7

(c) Show that the solution of the discretised problem is unique. 6

(d) Determine the linear system of equations resulting from the discretised problem where $f(x,y) = 1+xy$ and the boundary conditions are given by

$$u(x,0) = x(1-x), \qquad u(x,1) = 1,$$
$$u(0,y) = \frac{y}{2}(1+y), \qquad u(1,y) = y\ ,$$

using a stepsize of $h = \frac{1}{3}$ in both the x and y directions. 8

Useful Information

Half-range fourier **sine** series of period $2l$:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{\left(n - \frac{1}{2}\right)\pi x}{l}$$

$$\text{with} \quad B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{\left(n - \frac{1}{2}\right)\pi x}{l}\, dx \ .$$

Polar Coordinates:

$$x = r\cos\theta \ , \qquad y = r\sin\theta \ , \qquad 0 \le \theta \le 2\pi \ , \quad r \ge 0 \ .$$

Taylor Series in three variables:

$$f(x, y, z) = f(x_0, y_0, z_0) + \delta\mathbf{r} \cdot \nabla f|_{(x_0,y_0,z_0)} + O(|\delta\mathbf{r}|^2) \ ,$$

where $(h, k, l) = \delta\mathbf{r}$ with $h = x - x_0$, $k = y - y_0$ and $l = z - z_0$.