UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MA4006

SEMESTER: Spring 2007

MODULE TITLE: Engineering Mathematics 5

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. Patrick Johnson

PERCENTAGE OF TOTAL MARKS: 70%

INSTRUCTIONS TO CANDIDATES: **Answer any 4 questions correctly for full marks.**

1. (a) Consider the vector valued function

$$\mathbf{r}(t) = \begin{cases} (t,\, 1+t,\, 0) & \text{if } -1 \le t \le 0 \\ (\sin t,\, e^t,\, t^2) & \text{if } \; 0 \le t \le 1 \end{cases}$$

find its derivative, and hence find the unit tangent vector to the curve defined by $\mathbf{r}(t)$ and comment on its smoothness 10

(b) Sketch the curve $\mathbf{r}(t) = (t+1, t)$ where $-2 \le t \le 0$ in two dimensions and find an expression for its arclength from $t = -2$ to an arbitrary point on the curve. Hence find the arclength from $t = -2$ to $t = -1$. Write the equation for the curve in intrinsic form. 7

(c) A particle P moves on a disk towards the edge with position vector

$$\mathbf{r}(t) = (t^2 + t)\mathbf{b}$$

where $\mathbf{b}$ is a unit vector rotating together with the disk with constant angular speed ω in the counterclockwise sense. Find a form for the unit vector $\mathbf{b}$ and the acceleration of the particle P. 8

2. (a) Find the work done in moving a particle in a force field given by

$$\mathbf{F}(x, y, z) = 4x^2y\mathbf{i} - z\mathbf{j} + y^3xz\mathbf{k},$$

along the curve with parametric definition $\mathbf{r} = (t, t^2, 1)$ where $0 \le t \le 1$. 10

(b) Calculate the volume of a hemisphere using spherical coordinates of radius r. 6

(c) State Green's Theorem in the plane. Using Green's theorem evaluate the line integral $\oint \mathbf{F} \cdot d\mathbf{r}$ counterclockwise around the boundary of C of the region R, where $\mathbf{F} = x^2y\mathbf{i} - y^2x\mathbf{j}$ and $R : x^2 + y^2 \le 25$, $y \ge 0$. 9

3. (a) Find the directional derivative of the scalar valued function

$$f(x, y, z) = 3x^2y + z^3x - 4x$$

at the point $(1, -1, 0)$ in the direction of $2\mathbf{i} - \mathbf{j} + \mathbf{k}$. 7

(b) Using the fact that $\nabla\Omega$ (where $\Omega = \Omega(x, y)$ is a scalar function of position) is perpendicular to its own level curves, find a unit normal to the curve defined by $y - x^2 + 1 = 0$ at the point $(0, -1)$. Verify this result with a rough sketch. 7

(c) Show that the vector field $\mathbf{v} = -g\mathbf{k}$ (where g is the (constant) acceleration due to gravity) is a conservative vector field. Determine an associated *scalar potential* ϕ for this vector field $\mathbf{v}$. 5

(d) Use Taylor's theorem in two dimensions to find a first order approximation for $f(1, 0.5)$ based on quantities evaluated at the point $(1.5, 0.8)$ in the case where

$$f(x, y) = x^2y + y.$$

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4. (a) Explain, with the aid of a proof, the *linearity principle* (superposition principle) for a linear homogeneous partial differential equation. 4

(b) Consider the partial differential equation

$$A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} = f(x, y, u_x, u_y)$$

where A, B, C and f are all known functions. Explain the idea of *characteristics* of a partial differential equation. Noting that the equation defined above has characteristics defined by

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

suggest a way to classify equations as parabolic, elliptic or hyperbolic. Fully classify the following equations. Find and sketch the real characteristics (if any) of the following partial differential equations

$$u_{xx} + u_{xy} - 2u_{yy} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + x^2 = 0$$

$$u_{xx} + xu_{xy} + 2u = 0, \quad x > 0$$

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(c) Solve the following pseudo-differential equations for $u = u(x, y)$:

$$u_x - 2 = 0$$

$$u_x + u^2 = 0$$

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(d) Verify that $u(x, y) = \cos 3t \sin x$ is a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

for a suitable value of the constant c. Identify this value of c. 4

5. (a) The vibrations in an elastic string are governed by the one dimensional wave equation

$$u_{tt} = c^2 u_{xx}$$

where $u(x, t)$ is the deflection in the string. Consider the case where the string is fixed at the ends $x = 0$ and $x = l$. Let the initial deflection in the string be denoted by $u(x, 0) = f(x)$ and the initial velocity by $u_t(x, 0) = g(x)$. Find a solution of the wave equation to satisfy the boundary and initial conditions laid out above. 13

(b) Consider the equation

$$4u_{xx} - u_{yy} = 0$$

Using the transformation $\alpha = x + 2y$, $\beta = x - 2y$ reduce the equation to a simpler form and determine the particular solution that satisfies the boundary conditions

$$u(x, 0) = \sin x \quad u_y(x, 0) = 0$$

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6. Let Ω be the unit square $(0, 1) \times (0, 1)$, with boundary Γ and closure $\bar{\Omega} = [0, 1] \times [0, 1]$. Consider the boundary value problem

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega$$

with boundary conditions

$$u(x, y) = g(x, y), \quad (x, y) \in \Gamma$$

(a) Taking a fixed-width mesh h in both the x and y directions, formulate the discretised problem and outline a uniform mesh for this problem. The 2nd derivative can be approximated using the standard $O(h^2)$ operator. 4

(b) Show that the finite difference operator L^h satisfies a maximum principle of the form:

$$L^h v_{i,j} \geq 0 \;\forall\, i,j \;\Rightarrow\; \max_{\Omega^h} |v_{i,j}| \leq \max_{\Gamma^h} |v_{i,j}|$$

where $v_{i,j}$ is any mesh-function defined on $\bar{\Omega}^h$. 8

(c) Using the result from part (a) calculate the first two iterations, after your initial guess (use 0 as your initial guess), of a numerical solution to Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in the square $(0,2) \times (0,2)$ with boundary conditions

$$u(x,0) = x, \qquad u(x,1) = 2 - x,$$
$$u(0,y) = y, \qquad u(1,y) = 2 - y,$$

using a stepsize of $h = \frac{1}{4}$ in both the x and y directions. 13

Useful Information

Half-range fourier **cosine** series of period $2l$:

$$f(x) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}$$

$$\text{with} \quad A_n = 2/l \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx$$

Half-range fourier **sine** series of period $2l$:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

$$\text{with} \quad B_n = 2/l \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

Spherical Coordinates:

$$\begin{aligned} x &= r\cos\theta\sin\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\phi, \\ 0 \le \theta \le 2\pi, & \qquad 0 \le \phi \le \pi \end{aligned}$$

Taylor Series in more than one variable:

$$f(x,y,z) = f(x_0,y_0,z_0) + \delta\mathbf{r} \cdot \nabla f|_{(x_0,y_0,z_0)} + O(|\delta\mathbf{r}|^2)$$

where $(h,k,l) = \delta\mathbf{r}$ with $h = x - x_0$, $k = y - y_0$ and $l = z - z_0$.