Phase Diffusion Coefficient for Oscillators Perturbed by Colored Noise

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***Abstract*—The phase diffusion coefficient and the mean frequency of a two-dimensional nonlinear oscillator perturbed by colored noise is theoretically predicted and compared with numerical simulations of the Langevin system. At high oscillator frequencies, the first-order perturbation approximation of Demir is observed to yield inaccurate results for the phase diffusion coefficient when the spectrum of the noise sources decay faster than ω^{-2}. A novel asymptotic approach which describes the diffusion coefficient in such instances is developed.**

***Index Terms*—Demir's approximation, diffusion coefficient, Liouville equation, mean frequency, Monte Carlo simulation.**

I. Introduction

The problem of phase noise and its influence on oscillators has been the subject of much study in the last few decades [1]–[12]. Analytical and numerical work has predominantly focussed on white noise effects [2], [8]. However, recently, attention has turned to colored noise processes [3]. In recent work on the numerical simulation of a linearized oscillator, Gleeson [1] demonstrated a previously unreported anomalous scaling of the phase diffusion coefficient which is not predicted by industry standard asymptotic schemes. This work has highlighted the need for the development of second-order perturbation approximations which predict phase diffusion in self-sustained oscillators perturbed by colored noise sources. Such approximations are of interest to circuit designers as noisy effects induce unwanted phenomena such as timing jitter in clock circuits and interference in wireless communication systems. In this brief, we outline a stochastic approach based upon the Liouville equation [13], which yields the same result as Demir and co-workers [3], [8] to first order in the noise intensity, and also provides the correct second-order corrections to Demir's formula. Another prediction of this novel approximation procedure is that a noise induced shift of the mean frequency away from the deterministic carrier frequency will result. This has been numerically verified using Monte Carlo simulations. Finally, we derive asymptotic relations describing the behavior of the phase diffusion coefficient and the mean frequency at high frequencies, and show their dependence on the correlation function of the noise.

II. Oscillators Perturbed by Colored Noise

A two-dimensional self-sustained oscillator perturbed by independent Gaussian colored noises is considered

$$\dot{x} = f(x,y) + \xi(t) \tag{1}$$

$$\dot{y} = g(x,y) + \eta(t) \tag{2}$$

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where f and g are nonlinear functions, and $\xi(t)$ and $\eta(t)$ are the independent colored noise sources. The independent Gaussian colored noise processes have mean zero and variance κ^2 so their statistical behavior is fully characterized by their correlation functions $R(t)$

$$\langle \xi(t)\xi(t')\rangle = \langle \eta(t)\eta(t')\rangle = \kappa^2 R(t-t'). \tag{3}$$

The angular brackets are used throughout the paper to denote averaging over an ensemble. Alternatively, the noise may be described in terms of its spectrum, indeed this description of colored noise is particularly important for numerical simulations [1]

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\omega t)R(t)dt. \tag{4}$$

The phase dynamics of a near-circular oscillator's limit cycle is determined by transforming the Cartesian system to cylindrical polar coordinates: $x(t) = r(t)\cos(\theta(t))$, $y(t) = r(t)\sin(\theta(t))$. The phase angle for (1) and (2) is described by the following Langevin equation:

$$\frac{d\theta}{dt} = \Omega + \frac{\cos(\theta)\eta(t)}{r} - \frac{\sin(\theta)\xi(t)}{r}, \qquad \theta(0) = 0 \tag{5}$$

where Ω is the carrier frequency of the deterministic system. Equation (5) may be derived by transforming (1) and (2) using cylindrical polar coordinates and by applying the assumption that the limit cycle is strongly attracting. This work is to be specifically applied to oscillators with a near-circular limit cycle. We assume that the limit cycle of the oscillator is strongly attracting, so that fluctuations in r are negligible in comparison to phase fluctuations. By suitably scaling the Cartesian variables x and y we may set r to one. The result is a random differential equation describing the behavior of the phase on the limit cycle

$$\frac{d\theta}{dt} = \Omega + \cos(\theta)\eta(t) - \sin(\theta)\xi(t), \qquad \theta(0) = 0. \tag{6}$$

It is obvious that the random processes contribute to the phase equation in a nonlinear multiplicative fashion. Numerical simulations of this equation in [1] yielded some unforeseen results which we intend to verify here using analytic and asymptotic means. The analytical work will be discussed in relation to a specific correlation function

$$R(t) = \exp\left(-\frac{t^2}{2\tau^2}\right) \tag{7}$$

although the general method is applicable to any Gaussian stochastic process governed by an integrable correlation function.

The parameter τ in (7) characterizes the timescale of variation of the random process, therefore, τ^{-1} is an estimate of the bandwidth of the noise.

III. Analytical Approximations of Colored Noise Diffusion Coefficient

In this section, we give a detailed derivation of analytical approximations for the phase diffusion coefficient and the mean frequency. The nonlinear perturbation approximation of Demir [3] yields two formulas which are related to the mean frequency and the diffusion coefficient. These formulas for $\mu(t) = (\langle\theta\rangle - \Omega t)$ and the variance $\sigma^2(t) = \mathrm{var}(\theta)$ of the phase variable (see (13)) take the following form in our notation:

$$\frac{d\mu(t)}{dt} = \sum_{i=-\infty}^{\infty} ji|V_i|^2\kappa^2 \int_0^t R(t-s)\exp(j\Omega i(t-s))ds \quad (8)$$

$$\frac{d\sigma^2(t)}{dt} = \sum_{i=-\infty}^{\infty} 2|V_i|^2\kappa^2 \int_0^t R(t-s)\exp(j\Omega i(t-s))ds. \quad (9)$$

The quantity $|V_i|$ is determined from the Floquet vectors of the oscillator under discussion, and in this case we have employed similar techniques to the approach employed by Coram [7]. Coram's approach leads to the following results: $|V_{\pm 1}|^2 = 1/2$ with all other $V_i = 0$ for our example. For the oscillator under discussion (6), these equations(8) and (9) are equivalent to the following forms:

$$\frac{d\langle\theta\rangle}{dt} = \Omega - \kappa^2 \int_0^t R(t-s)\sin(\Omega(t-s))ds + O(\kappa^4) \quad (10)$$

$$\frac{d(\mathrm{var}(\theta))}{dt} = 2\kappa^2 \int_0^t R(t-s)\cos(\Omega(t-s))ds + O(\kappa^4). \quad (11)$$

It is these formulas which we intend to benchmark in this paper. One final point to note is the concept of the accuracy of these formulas to first and second order in the noise intensity. Accuracy to first order implies the inclusion of all κ^2 contributions whilst "second-order accuracy" implies the inclusion of all contributions up to and including κ^4. It will be shown later that first-order contributions correspond to single integrals of the correlation function whilst second-order contributions correspond to triple integrals of the correlation function. It is for this reason that we will refer to Demir's approach as a first-order approximation.

Gleeson [1] investigated the phase diffusion coefficient of the linearized oscillator (6) using Monte Carlo simulations. This method involves approximating the continuous Gaussian noise processes by a large number of Fourier modes whose coefficients are dependent on the statistical characteristics of the noise [1]. The problem of solving a Langevin equation is thus reduced to one involving the numerical solution of an ordinary differential equation. Each time we solve the differential equation we get a realization of the phase variable. By computing a large number of realizations we have an approximate ensemble which may be analyzed to derive the statistical behavior of the phase variable.

The large time growth of the mean and the variance of the phase variable is linear in time and we may define these quantities asymptotically as follows:

$$\langle\theta\rangle \sim \omega t \text{ as } t \to \infty \quad (12)$$

where $\langle\theta\rangle$ is the mean of the phase variable θ at time t. The time derivative of $\langle\theta\rangle$ as $t \to \infty$ is ω, the mean frequency. Meanwhile the variance of the phase variable at time t obeys the following relation:

$$\mathrm{var}(\theta) \sim Dt \text{ as } t \to \infty. \quad (13)$$

The time derivative of $\mathrm{var}(\theta)$ as $t \to \infty$ is the phase diffusion coefficient D. The phase variable is numerically simulated according to (6) for large times. The resulting time dependent moments of θ are fitted to linear plots after neglecting some initial data, the quantities D and ω being derived from the slopes of the fitted lines. A fuller and more detailed explanation of the method may be found in Gleeson [1]; however, in this brief, we will be concentrate on theoretical predictions of numerical results.

We begin by changing the frame of reference of our Langevin system (6) by applying the following transformation:

$$x(t) = \theta - \Omega t. \quad (14)$$

The new variable $x(t)$ obeys the following equation:

$$\frac{dx}{dt} = F(t)\cos(x) - G(t)\sin(x), \qquad x(0) = 0 \quad (15)$$

where

$$F(t) = \eta(t)\cos(\Omega t) - \xi(t)\sin(\Omega t) \quad (16)$$

$$G(t) = \eta(t)\sin(\Omega t) + \xi(t)\cos(\Omega t). \quad (17)$$

This new equation (15) follows from (6) by differentiating (14). The stochastic components of (15) have mean zero and the correlation functions

$$\langle F(t)F(t')\rangle = \langle G(t)G(t')\rangle = R(t-t')\cos(\Omega(t-t')) \quad (18)$$

$$\langle G(t)F(t')\rangle = -\langle F(t)G(t')\rangle = R(t-t')\sin(\Omega(t-t')). \quad (19)$$

The density $\rho(x,t)$ describes the behavior of the ensemble of solution trajectories $x(t)$ as time evolves and this density obeys the stochastic Liouville equation [13]. This equation is an expression of the conservation of the number of systems in the ensemble

$$\frac{\partial\rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}[[F(t)\cos(x) - G(t)\sin(x)]\rho(x,t)] \quad (20)$$

with initial condition

$$\varrho(t=0) = \delta(x). \quad (21)$$

The passage from (20) to a form which will yield a useful and accurate analytic approximation of the phase diffusion coefficient is well known [14] and involves Fourier transforming the spatial variable of (20) as follows:

$$\tilde{\rho}(k,t) = \int_{-\infty}^{\infty} dx\rho(x,t)\exp(ikx). \quad (22)$$

The Liouville equation (20) is Fourier transformed by multiplying the equation by $\exp(ikx)$ and integrating by parts. The exponential definitions of cosine and sine lead to $\tilde{\rho}(k+1,t)$

and $\tilde{\rho}(k-1,t)$ contributions in (23) below. If we integrate the Fourier transformed Liouville equation the stochastic partial differential equation is reduced to a stochastic integral equation of the form

$$\tilde{\rho}(k,t) = 1 + k\int_0^t (\tilde{\rho}(k+1,u)H(u) + \tilde{\rho}(k-1,u)L(u))du \tag{23}$$

where the stochastic terms

$$H(t) = i\frac{F(t)}{2} - \frac{G(t)}{2} \tag{24}$$

$$L(t) = i\frac{F(t)}{2} + \frac{G(t)}{2} \tag{25}$$

have mean zero and the following correlation behavior:

$$\langle H(t)H(t')\rangle = \langle L(t)L(t')\rangle = 0 \tag{26}$$

$$\langle L(t)H(t')\rangle = -\frac{R(t-t')}{2}\exp(i\Omega(t-t')) \tag{27}$$

$$\langle H(t)L(t')\rangle = -\frac{R(t-t')}{2}\exp(-i\Omega(t-t')). \tag{28}$$

By averaging $\tilde{\rho}(k,t)$ over the realizations of $H(t)$ and $L(t)$ the characteristic function of the probability density $\langle\rho(x,t)\rangle = P(x,t)$ is obtained. By differentiating the characteristic function we obtain representations of the moments of the phase variable x

$$\frac{1}{i}\frac{\partial\langle\tilde{\rho}(k,t)\rangle}{\partial k}|_{k=0} = \langle x\rangle \tag{29}$$

$$-\frac{\partial^2\langle\tilde{\rho}(k,t)\rangle}{\partial k^2}|_{k=0} = \langle x^2\rangle. \tag{30}$$

The results (29) and (30) may be related to the moments of the phase variable θ as follows:

$$\langle\theta\rangle = \Omega t + \langle x\rangle \tag{31}$$

$$\langle\theta^2\rangle = \langle x^2\rangle + 2\Omega t\langle x\rangle + \Omega^2 t^2. \tag{32}$$

Approximations to the characteristic function $C_x(k,t)$ may be obtained by successively iterating the integral (23), and averaging over the ensemble. The iteration of (23) involves substituting representations for $\tilde{\rho}(k+1,u)$ and $\tilde{\rho}(k-1,u)$ which are defined using (23). The stochastic components of the integral (23) are Gaussian and have mean zero. The Gaussian assumption means that all even moments may be rewritten in terms of the second central moment. In terms of the problem under consideration we are concerned only with terms up to and including the fourth-order moments. To this end, we may apply the general identity

$$\begin{aligned}&\langle A(u)B(v)C(w)D(x)\rangle\\ =&\langle A(u)B(v)\rangle\langle C(w)D(x)\rangle + \langle A(u)C(w)\rangle\langle B(v)D(x)\rangle\\ &+ \langle A(u)D(x)\rangle\langle B(v)C(w)\rangle.\end{aligned} \tag{33}$$

This idea is explained in greater detail in Van Kampen [13]. Iterating two times and averaging gives a result accurate to first order in the noise intensity κ^2

$$\langle\tilde{\rho}(k,t)\rangle = 1 - \kappa^2 a(k,t) - \kappa^2 b(k,t) + O(\kappa^4) \tag{34}$$

where

$$a(k,t) = \frac{k(k+1)}{2}\int_0^t du\int_0^u dvR(u-v)\exp(i\Omega(u-v)) \tag{35}$$

$$b(k,t) = \frac{k(k-1)}{2}\int_0^t du\int_0^u dvR(u-v)\exp(-i\Omega(u-v)). \tag{36}$$

Hence, by differentiating the first-order approximation of the characteristic function (34) with respect to k and to t the following analytical results may be obtained using (31) and (32):

$$\frac{d\langle\theta\rangle}{dt} = \Omega - \kappa^2\int_0^t duR(u)\sin(\Omega u) + O(\kappa^4) \tag{37}$$

$$\frac{d(\mathrm{var}\theta)}{dt} = 2\kappa^2\int_0^t duR(u)\cos(\Omega u) + O(\kappa^4). \tag{38}$$

Taking the limit of (38) as $t\to\infty$, we get a first-order approximation of the diffusion coefficient. This is identical to Demir's result [3] for this particular oscillator

$$D = 2\pi\kappa^2 S(\Omega) + O(\kappa^4). \tag{39}$$

The analytical approximation also predicts a shift away from the deterministic carrier frequency induced by the nonlinear interactions with the noise as shown in (37). This induced shift away from the carrier frequency although not explicitly mentioned by Demir [3] is implicitly implied from the formula he derives for the mean frequency. Demir's formula is correct to first order in the noise intensity. This phenomenon is likely to be important particularly when dealing with the power spectral density.

If we iterate (23) further to account for second-order effects, we obtain improved representations for the time derivative of $\langle\theta\rangle$

$$\begin{aligned}\frac{d\langle\theta\rangle}{dt} =& \Omega - \kappa^2\int_0^t duR(u)\sin(\Omega u)\\ &+ \kappa^4\int_0^t du\int_0^{t-u} dv\int_0^{t-u-v} dwQ(u+v,v+w)\\ &\times\sin(\Omega(u+2v+w))\\ &+ \kappa^4\int_0^t du\int_0^{t-u} dv\int_0^{t-u-v} dwQ(u+v+w,v)\\ &\times\sin(\Omega(u+2v+w))\\ &+ O(\kappa^6)\end{aligned} \tag{40}$$

where

$$Q(x,y) = R(x)R(y). \tag{41}$$

Similarly a representation for the time derivative of the variance may be derived

$$\begin{aligned}&\frac{d(\mathrm{var}(\theta))}{dt}\\ &= 2\kappa^2\int_0^t duR(u)\cos(\Omega u)\\ &- 2\kappa^4\int_0^t du\int_0^{t-u} dv\int_0^{t-u-v} dwQ(u+v+w,v)\\ &\times\cos(\Omega(u+w))\\ &- 4\kappa^4\int_0^t du\int_0^{t-u} dv\int_0^{t-u-v} dwQ(u+v,v+w)\\ &\cos(\Omega(u+2v+w))\\ &- 4\kappa^4\int_0^t du\int_0^{t-u} dv\int_0^{t-u-v} dwQ(u+v+w,v)\\ &\times\cos(\Omega(u+2v+w))\\ &+ O(\kappa^6).\end{aligned} \tag{42}$$

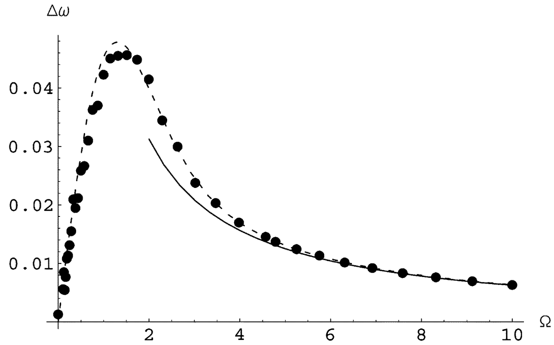

Fig. 1. Frequency deviation $\Delta\omega$ as a function of the oscillator frequency Ω, with noise level $\kappa = .25\tau^{-1}$, calculated from the phase mean over 10^4 realizations. The correlation function of the noise sources is given by (7). The dotted line is determined from the approximation (38); the line is the asymptotic power law (46). The frequency Ω and the mean frequency are expressed in units of τ^{-1}.

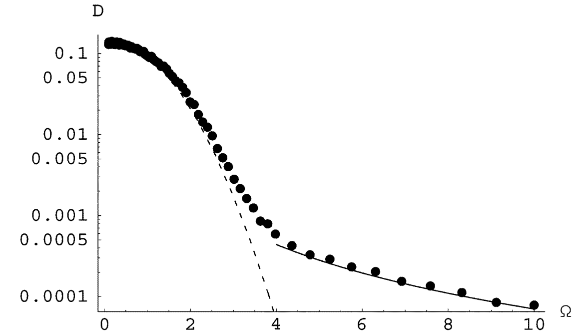

Fig. 2. Phase diffusion D as a function of the oscillator frequency Ω, with noise level $\kappa = .25\tau^{-1}$, calculated from the phase variance over 10^4 realizations. The correlation function of the noise sources is given by (7). The dotted line shows the approximation; the line is the asymptotic power law for large Ω, $D \sim \sqrt{\pi}\tau\kappa^4\Omega^{-2}$. The frequency Ω and the diffusion coefficient are expressed in units of τ^{-1}. It should be noted that $T_{\text{eff}} = \sqrt{\pi}\tau$.

Representations for the mean frequency and the diffusion coefficient which include second-order effects may be determined by taking the limit of (40) and (42) as $t \to \infty$.

IV. Asymptotic Approximations

In this section we calculate an analytical result which describes the scaling of the diffusion coefficient and the mean frequency for our linearized phase (6). The results will apply for large frequencies beyond the bandwidth of the continuous noise process where for certain noise correlation functions second-order effects dominate. This divergence occurs in the high-frequency region

$$\Omega \gg \tau^{-1}. \tag{43}$$

To begin with we note the following asymptotic results which may be derived using repeated integration by parts

$$\int_0^\infty f(t)\cos(\Omega t)dt \sim -f'(0)\Omega^{-2} + f'''(0)\Omega^{-4} - \cdots \quad \text{as } \Omega \to \infty \tag{44}$$

$$\int_0^\infty f(t)\sin(\Omega t)dt \sim f(0)\Omega^{-1} - f''(0)\Omega^{-3} - \cdots \quad \text{as } \Omega \to \infty. \tag{45}$$

Thus, we look for contributions of this form when we derive the coefficient of Ω^{-n} in our asymptotic expansions. This idea is first applied to the asymptotic analysis of the mean frequency equation (40). The resulting asymptotic form is

$$\Delta\omega \sim \kappa^2 R(0)\Omega^{-1} - \kappa^2 R''(0)\Omega^{-3} \text{ as } \Omega \to \infty. \tag{46}$$

For the correlation function R, the frequency deviation $\Delta\omega = \Omega - \omega$ has been plotted in Fig. 1 for various values of Ω. We note that under assumption 4.1 of Demir [3] the nonlinear perturbation approximation (8) predicts that the frequency deviation will be zero. This is contradicted by our numerical and analytical work. This verifies that it is incorrect to take account of only $|V_0|$ in (8), as in Demir's Assumption 4.1—the higher harmonics have important effects in this case. Indeed we observe a previously unreported behavior which is induced by the multiplicative noise processes, and have derived an accurate asymptotic approximation which predicts the behavior of the mean frequency beyond the noise bandwidth $\Omega \gg \tau^{-1}$.

By similarly expanding the trigonometric elements of the integrals representing D we find that

$$D \sim \left(2\kappa^4 \int_0^\infty R^2(u)du - 2\kappa^2 R'(0)\right)\Omega^{-2} \text{ as } \Omega \to \infty \tag{47}$$

where the second term comes from (39). One can see that if $R'(0) = 0$, then

$$D \sim T_{\text{eff}}\kappa^4\Omega^{-2} \text{ as } \Omega \to \infty \tag{48}$$

where

$$T_{\text{eff}} = 2\int_0^\infty R^2(u)du = 2\pi\int_{-\infty}^\infty S^2(u)du. \tag{49}$$

This condition corresponds to a noise source whose spectrum decays faster than ω^{-2}. The diffusion coefficient for the correlation function $R(t)$ of (7) has been numerically simulated and plotted as a function of Ω in Fig. 2. This correlation function obeys the condition $R'(0) = 0$ and thus has a continuous noise spectrum which decays faster than ω^{-2}

$$S(\omega) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\omega^2\tau^2}{2}\right). \tag{50}$$

The first-order approximation (39) is plotted and compared to the Monte Carlo simulations and proves to be accurate for moderate values of the carrier frequency Ω. However for larger values Ω, particularly when $\Omega \gg \tau^{-1}$, Demir's result (42) that D becomes exponentially small proves to be inaccurate. The line in Fig. 2 displays the accuracy of the asymptotic result (48) for large frequencies, a result which is generic for continuous noise processes whose spectra decay faster than ω^{-2}. If $R'(0) \neq 0$

$$D \sim (-2\kappa^2 R'(0))\Omega^{-2} + O(\kappa^4) \text{ as } \Omega \to \infty. \tag{51}$$

Gleeson [1] has demonstrated numerically these asymptotic decays for various correlation functions.

V. Discussion

The focus of this letter has been the development of a second-order perturbation expansion which captures the unusual behavior of the diffusion coefficient at high frequencies for certain correlation functions. The asymptotic work shows that when the noise spectrum decays faster than ω^{-2} the falloff of the diffusion coefficient obeys the relation

$$D \sim T_{\text{eff}} \kappa^4 \Omega^{-2} \tag{52}$$

for large Ω, with T_{eff} calculated from the following identity:

$$T_{\text{eff}} = 2 \int_0^{\infty} R^2(u)du = 2\pi \int_{-\infty}^{\infty} S^2(u)du. \tag{53}$$

The existence of such scaling has previously been predicted by Gleeson [1] and here we have derived a result for the scaling constant T_{eff}. This asymptotic behavior has been demonstrated in Fig. 2 where the calculated value T_{eff} is $\sqrt{\pi}\tau$. This relationship between the correlation function and T_{eff} is nontrivial and yields results which closely agree with the fitted numerical results of Gleeson [1]. Thus, this work enables the prediction of second-order noise effects from the correlation function and hence accurate analysis of the diffusion coefficient at high frequencies. Another feature which we have demonstrated is the shift away from the deterministic carrier frequency induced by the multiplicative noise sources. This result has been verified both numerically using Monte Carlo simulations in Fig. 1 and analytically using our stochastic Liouville equation approximation and is accurate to second order in the noise intensity. All our results have been calculated with regard to a simple phase oscillator (6), however similar phenomena are expected in oscillators with the same multiplicative noise structure.

The state dependent modulation of noise sources which is the most general form of this problem has been considered by Demir [3]. In this work the effect of additive Gaussian colored noises on an oscillator defined in Cartesian coordinates is analyzed by linearizing the oscillator about its deterministic limit cycle. The approach may be generalized to examples of state dependent modulation of noise sources provided we abide by the assumption that the amplitude is strongly attracting and Fourier expand the resulting system. The phase equation will have the form [compare to (6)]

$$\frac{d\theta}{dt} = \sum_{i=-\infty}^{\infty} w_i(t) \exp(ji\theta) \tag{54}$$

where $w_i(t)$ contain the stochastic components of the problem. In principle, a derivation of the mean frequency and the diffusion coefficient to order κ^4 will follow in a similar fashion as before from the Liouville equation. It is important to note that this paper is focussed primarily on second-order noise effects and their consequences even for a simple, single-harmonic oscillator such as (6).

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