Mathematical geoscience

1. Show that $\int_{\frown} \cos\theta \, d\omega = \pi$, and deduce that $E_{b\nu} = \pi B_\nu$, where $E_{b\nu}$ is the black body radiation emitted normally from a surface, per unit area.

   Use Planck's law
$$B_\nu(T) = \frac{2h\nu^3}{c^2[e^{h\nu/kT} - 1]}$$
   to derive the Stefan-Boltzmann law in the form
$$E = \int_0^\infty E_{b\nu}\, d\nu = \sigma T^4,$$
   where
$$\sigma = \frac{2\pi k^4}{c^2 h^3} \int_0^\infty \frac{z^3\, dz}{e^z - 1}.$$
   By evaluating the integral and using the values $c = 2.998 \times 10^8$ m s^{-1}, $k = 1.381 \times 10^{-23}$ J K^{-1}, $h = 6.626 \times 10^{-34}$ J s, evaluate the Stefan Boltzmann constant σ.

   Hint:
$$\sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. A wet adiabat is calculated from the isentropic equation
$$\rho_a c_p \frac{dT}{dz} - \frac{dp}{dz} + \rho_a L \frac{dm}{dz} = 0,$$
   where
$$m = \frac{\rho_v}{\rho_a}, \quad p = \frac{\rho_a RT}{M_a}, \quad p_{SV} = \frac{\rho_v RT}{M_v},$$
   and
$$\frac{dp_{SV}}{dT} = \frac{\rho_v L}{T}, \quad \frac{dp}{dz} = -\rho_a g.$$
   Deduce that T and p_{SV} can be calculated from the equations
$$\begin{aligned} \frac{dT}{dz} &= -\Gamma_w(\rho_v, p, T), \\ \frac{dp_{SV}}{dz} &= -\frac{\rho_v L}{T}\Gamma_w, \end{aligned}$$
   where $\rho_v = \rho_v(p_{SV}, T)$, and Γ_w should be determined. Using values $M_v/M_a = 0.62$, $L = 2.5 \times 10^6$ J kg^{-1}, $T = 290$ K, $c_p = 10^3$ J kg^{-1} K^{-1}, $\rho_v = 0.01$ kg m^{-3}, $p = 10^5$ Pa, $g = 10$ m s^{-2}, $\rho_a = 1$ kg m^{-3}, show that a typical value of Γ_w is 6 K km^{-1}.

By assuming that $T \approx$ constant (why?), derive a differential equation for p_{SV} as a function of z in terms of two dimensionless coefficients

$$a = \frac{M_v L}{RT}, \quad \beta = \frac{M_v}{M_a}\frac{L}{c_p T},$$

and estimate their values (you will need also the values $M_v = 18 \times 10^{-3}$ kg mole^{-1}, $R = 8.3$ J mole^{-1} K^{-1}). Derive from this an autonomous differential equation for the *molar specific humidity* $h = p_{SV}/p$. Assuming a surface value of $h \approx 0.02$, show that $H = \beta a h \sim O(1)$, and by writing $z = RTZ/M_a g$, show that

$$\frac{dH}{dZ} = -\frac{(\beta - 1)H}{1 + H}.$$

Deduce that for $Z \sim O(1)$,

$$H \approx H_0 \exp\left[H_0 - (\beta - 1)Z\right]:$$

humidity decreases rapidly with altitude.

3. The temperature T, CO_2 pressure p, and planetary albedo a satisfy the ordinary differential equations

$$\begin{aligned} c\dot{T} &= \tfrac{1}{4}Q(1-a) - \sigma\gamma T^4, \\ t_i \dot{a} &= a_{\text{eq}}(T) - a, \\ \frac{M_{\text{CO}_2} A}{M_a g}\dot{p} &= -A_L W + v, \end{aligned}$$

where

$$a_{\text{eq}}(T) = a_1 - \tfrac{1}{2}a_2\left[1 + \tanh\left(\frac{T - T^*}{\Delta T}\right)\right],$$

where $a_1 = 0.58$, $a_2 = 0.47$, $T^* = 283$ K, $\Delta T = 24$ K,

$$W = W_0\left(\frac{p}{p_0}\right)^{\mu} \exp\left[\frac{T - T_0}{\Delta T_c}\right],$$

and

$$\gamma(p) = \gamma_0 - \gamma_1 p.$$

Show how to non-dimensionalise the system to the dimensionless form

$$\begin{aligned} \varepsilon\dot{\theta} &= 1 - a - (1 - a_0)\left(1 + \tfrac{1}{4}\nu\theta\right)^4 (1 - \nu\lambda p), \\ \dot{a} &= B(\theta) - a, \\ \dot{p} &= \alpha\left[1 - wp^{\mu}e^{\theta}\right], \end{aligned}$$

and show that

$$\alpha = \frac{vgt_i M_a}{Ap_0 M_{\text{CO}_2}}, \quad w = \frac{A_L W_0}{v}, \quad \varepsilon = \frac{4c\Delta T_c}{t_i Q}, \quad \nu = \frac{4\Delta T_c}{T_0}, \quad \lambda = \frac{\gamma_1 p_0}{\nu\gamma_0}.$$

What is the function $B(\theta)$? What is the definition of a_0?

Using the values $v = 3 \times 10^{11}$ kg y^{-1}, $g = 9.81$ m s^{-2}, $M_a = 28.8 \times 10^{-3}$ kg mole^{-1}, $M_{CO_2} = 44 \times 10^{-3}$ kg mole^{-1}, $t_i = 10^4$ y, $A = 5.1 \times 10^{14}$ m^2, $p_0 = 36$ Pa, $A_L = 1.5 \times 10^{14}$ m^2, $W_0 = 2 \times 10^{-3}$ kg m^{-2} y^{-1}, $c = 10^7$ J m^{-2} K^{-1}, $Q = 1370$ W m^{-2}, $\Delta T_c = 13$ K, $T_0 = 288$ K, $\gamma_0 = 0.64$, $\gamma_1 = 0.8 \times 10^{-3}$ Pa^{-1}, $\mu = 0.3$, show that

$$\alpha \approx 1.05, \quad w \approx 1, \quad \varepsilon \approx 1.2 \times 10^{-6}, \quad \nu \approx 0.18, \quad \lambda \approx 0.25,$$

and find the value of a_0, assuming $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4}.

Hence show that θ rapidly approaches a quasi-steady state given by

$$\theta \approx \Theta(a, p) = \kappa(a_0 - a) + \lambda p,$$

where

$$\kappa = \frac{1}{\nu(1 - a_0)}.$$

In the phase plane of a and p satisfying

$$\begin{aligned} \dot{a} &= B(\Theta) - a, \\ \dot{p} &= \alpha\left[1 - wp^{\mu}e^{\Theta}\right], \end{aligned} \qquad (*)$$

show that the p nullcline is a monotonically increasing function $a_p(p)$ of p, and that the a nullcline is a monotonically decreasing function $a_a(p)$ of p, providing $-B'(\theta) < \nu(1 - a_0)$ for all θ. Show conversely that if there is a range of θ for which $-B'(\theta) > \nu(1 - a_0)$, then the a nullcline is multivalued.

Suppose that the a nullcline is indeed multivalued, and that there is always a unique steady state. Show that at low, intermediate and high values of w, this equilibrium can lie on the lower, intermediate or upper branch of the a nullcline.

By consideration from the phase plane of the signs of the partial derivatives of the right hand sides of $(*)$ (and without detailed calculation), show that when they exist, the upper and lower branch steady states are stable, but that the intermediate steady state will be oscillatorily unstable if α is small enough.

How would you expect the solutions to behave if $\alpha \ll 1$?

4. Calcium carbonate, $CaCO_3$, in the form of calcite or aragonite, dissolves in acid to form calcium and bicarbonate ions according to the reaction

$$(CaCO_3 +)\ H^+ \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} Ca^{2+} + HCO_3^-.$$

In addition, the bicarbonate buffering system is described by the reactions

$$\begin{aligned} (H_2O +)\ CO_2 &\underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} HCO_3^- + H^+, \\ HCO_3^- &\underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} CO_3^{2-} + H^+, \end{aligned}$$

where the brackets on H_2O and $CaCO_3$ indicate that these substances are present in unlimited supply, and are thus ignored in writing the rate equations.

Write down the rate equations for the reactant concentrations $[H^+]$, $[Ca^{2+}]$, $[HCO_3^-]$, $[CO_2]$ and $[CO_3^{2-}]$, and by assuming equilibrium, derive three equations for the concentrations in terms of the equilibrium constants

$$K_1 = \frac{k_1}{k_{-1}}, \quad K_2 = \frac{k_2}{k_{-2}}, \quad K_3 = \frac{k_3}{k_{-3}},$$

and by suitable summation of the equations, derive the additional relations

$$[HCO_3^-] - [Ca^{2+}] + [CO_2] + [CO_3^{2-}] = P,$$

$$[H^+] + 2[Ca^{2+}] - 2[CO_3^{2-}] - [HCO_3^-] = Q,$$

where P and Q are constants.

Define the dissolved inorganic carbon C to be

$$C = [HCO_3^-] + [CO_2] + [CO_3^{2-}],$$

and the alkalinity to be

$$A = [HCO_3^-] + 2[CO_3^{2-}].$$

By writing

$$\xi = \frac{[H^+]}{C}, \quad \eta = \frac{[CO_2]}{C}, \quad p = \frac{[Ca^{2+}]}{C}, \quad \lambda_i = \frac{K_i}{C},$$

show that

$$\begin{aligned} \xi + 2p &= q + \alpha, \\ \frac{\lambda_1 \eta}{\xi^2}(\xi + 2\lambda_2) &= \alpha, \\ \frac{\lambda_3 \xi^2}{\lambda_1 \eta} &= p, \\ \eta &= \frac{1}{1 + \dfrac{\lambda_1}{\xi} + \dfrac{\lambda_1 \lambda_2}{\xi^2}}, \end{aligned}$$

where

$$P = C(1-p), \quad Q = Cq, \quad A = C\alpha.$$

If all the dissolved carbon is formed from calcium carbonate, we may suppose $P = 0$, and if the system is charge neutral, we may take $Q = 0$. Show in this case that ξ satisfies the two equations

$$\xi + 2 = \frac{\lambda_1(\xi + 2\lambda_2)}{\xi^2 + \lambda_1 \xi + \lambda_1 \lambda_2} = \lambda_3(\xi + 2\lambda_2).$$

(*The extra equation occurs because C is not known.*) Show that an exact solution of this pair of equations occurs for $\xi = 0$, $\lambda_2\lambda_3 = 1$, and deduce that the dissolved carbon concentration is

$$C = \sqrt{K_2 K_3}.$$

Using the values $K_1 = 1.4 \times 10^{-6}$ mol kg^{-1}, $K_2 = 1.07 \times 10^{-9}$ mol kg^{-1}, $K_2K_3 = 1.6 \times 10^{-8}$ mol^2 kg^{-2}, show that this implies that $C \approx 0.126 \times 10^{-3}$ mol kg^{-1}, which is about sixteen times lower than the observed value. The discrepancy may be ascribed to the presence of many other ionic species, and the presence of other carbonate reactions, so that the assumptions $P = 0$, $Q = 0$ are invalid. Instead we will take the observed values for DIC of $C = 2 \times 10^{-3}$ mol kg^{-1}, and for carbonate alkalinity $A = 2.3 \times 10^{-3}$ mol kg^{-1}. Show in this case that $\alpha = 1.15$, and that

$$\lambda_2 \ll \lambda_1 \ll 1 \ll \lambda_3.$$

By anticipating that $\lambda_2 \lesssim \xi \ll \lambda_1$, show that

$$\xi \approx \left(\frac{2-\alpha}{\alpha-1}\right)\lambda_2, \quad \eta \approx \frac{(2-\alpha)^2\lambda_2}{(\alpha-1)\lambda_1},$$

and deduce that $\xi \approx 0.3 \times 10^{-5}$, $\eta \approx 0.36 \times 10^{-2}$, and that $\mathrm{pH} = -\log_{10}[\mathrm{H}^+] \approx 8.2$, as observed.

Show that the observed concentration of $[\mathrm{Ca}^{2+}] \approx 0.9 \times 10^{-2}$ mol kg^{-1} implies that $p \approx 4.5$, and that then $q \approx 7.85$. Show also that this value of p requires that $\lambda_3 \approx 1.26 \times 10^6$, and thus that $K_2K_3 = 2.7 \times 10^{-6}$ mol^2 kg^{-2}, as opposed to the value quoted above.

5. Find a relationship between the hydraulic radius R and the area A for triangular (notch shaped) or wide rectangular (canal shaped) cross sections. Hence show that Chézy's and Manning's laws both lead to a general relationship of the form

$$Q = \frac{cA^{m+1}}{m+1},$$

with $0 < m < 1$, giving explicit prescriptions for c and m. For a canal of depth h, show that the flow is turbulent if

$$h \gtrsim 10^2 \nu^{2/3} \left(\frac{f}{Sg}\right)^{1/3},$$

where ν is the kinematic viscosity, f is the friction factor, S is the slope and g is gravity. Taking $\nu = 10^{-6}$ m^2 s^{-1}, $f = 0.01$, $S = 10^{-3}$, $g = 10$ m s^{-2}, find a critical depth for turbulence. Is the Shannon turbulent?

6. The cross sectional area of a river A is assumed to satisfy the wave equation

$$\frac{\partial A}{\partial t} + cA^m \frac{\partial A}{\partial s} = 0,$$

where s is distance downstream. Explain how this equation can be derived from the principle of conservation of mass. What assumptions does your derivation use?

A river admits a steady discharge $Q = Q_+$. At $t = 0$, a tributary at $s = 0$ is blocked, causing a sudden drop in discharge to $Q_- < Q_+$. Solve the equation for A using a characteristic diagram and show that an *expansion fan* branches from $s = 0$, $t = 0$. What is the hydrograph record at a downstream station $s = s_0 > 0$?

Later, the tributary is re-opened, causing a sudden rise from Q_- to Q_+. Draw the characteristic diagram, and show that a shock wave propagates forwards. What is its speed?

7. Why should the equation

$$A_t + cA^m A_s = M$$

represent a better model of slowly varying river flow than that with $M = 0$? Find the general solution of the equation, given that $A = 0$ at $s = 0$, and $A = A_0(s)$ at $t = 0$, $s > 0$, assuming $M = M(s)$. Find also the steady state solution $A_{eq}(s)$. How would you expect solutions representing disturbances to this steady profile to behave?

Suppose now that M is constant, and $A_0 = A_{eq} + \overline{A}\delta(s)$, representing an initial flood concentrated at $s = 0$.[1] Show that the resulting flood occurs in $s_- < s < s_+$, and show that the profile of A between s_- and s_+ is given implicitly by

$$A^{m+1} - (A - Mt)^{m+1} = \frac{(m+1)Ms}{c},$$

and deduce that

$$s_- = \frac{cM^m t^{m+1}}{(m+1)}.$$

What happens as $M \to 0$?

8. Derive the St. Venant equations from first principles, indicating what assumptions you make concerning the channel cross section. Derive a non-dimensional form of these equations assuming Manning's roughness law and a triangular cross section. [*Assume that there is no source term in the equation of mass conservation.*]

[1] $\delta(s)$ is the delta function, defined by $\delta(s) = 0$ for $s \neq 0$, and infinite at $s = 0$ in such a way that $\int_{-\infty}^{\infty} \delta(s)\,ds = 1$. The delta function is a *generalised function*, and can be thought of a sharp spiked function which is the limit of the Gaussian $\sqrt{\varepsilon\pi}\exp\left(-\frac{x^2}{\varepsilon}\right)$ as $\varepsilon \to 0$.

A sluice gate is opened at $s = 0$ so that the discharge there increases from Q_- to Q_+. The hydrograph is measured at $s = l$. Using l as a length scale, and with a corresponding time scale $\sim l/u$, derive an approximate expression for the dimensionless discharge in terms of A, if the Froude number is small, and also $\varepsilon = [\bar{h}]/Sl \ll 1$, where $[\bar{h}]$ is the scale for the mean depth and S is the slope.

Hence show that A satisfies the approximate equation

$$\frac{\partial A}{\partial t} + \tfrac{4}{3}A^{1/3}\frac{\partial A}{\partial s} = \tfrac{1}{4}\varepsilon\frac{\partial}{\partial s}\left[A^{5/6}\frac{\partial A}{\partial s}\right].$$

What do you think the difference between the hydrographs for $\varepsilon = 0$ and $0 < \varepsilon \ll 1$ might be?

9. A dimensionless long wave model for slowly varying flow of a river of depth h and mean velocity u is given in the form

$$h_t + (uh)_s = M(s),$$

$$0 = 1 - \frac{u^2}{h} - \varepsilon h_s,$$

where $\varepsilon \ll 1$.

How would you physically interpret the positive source term $M(s)$?

Show that for small ε, the model can be reduced to the approximate form

$$h_t + (h^{3/2})_s = M(s) + \tfrac{1}{2}\varepsilon[h^{3/2}h_s]_s.$$

Show that if $h = 0$ at $s = 0$, then an approximate steady state solution is given by

$$h = \left\{\int_0^s M(s)\,ds\right\}^{2/3}. \quad (*)$$

Find this approximate solution if $M \equiv 1$. Can you find a function $M(s)$ for which $(*)$ is an exact solution?

Explain why the condition of a horizontal water surface might be an appropriate boundary condition to apply at $s = 1$, and show that in terms of the scaled variables, this implies $h_s = 1/\varepsilon$ at $s = 1$. Show that with this added boundary condition, the approximate solution (when $M \equiv 1$) is still appropriate, except in a boundary layer near the outlet.

Next, suppose that $M = 0$ for large enough s, and that $\int_0^\infty M(s)\,ds = 1$. Write down the linear equation satisfied by small perturbations H to the steady state $h = 1$ when s is large.

By seeking solutions of the form $\exp[\sigma t + iks]$, show that small wave-like disturbances travel at speed $\frac{3}{2}$ and decay on a time scale $t \sim O(1/\varepsilon)$.

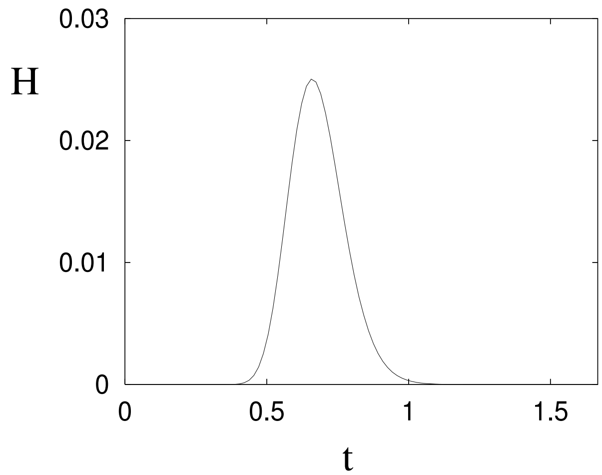

Figure 1: $H(s,t)$ plotted at fixed $s = 1$ as a function of t, using values $\varepsilon = 0.03$, $l = 0.005$, $\delta = 1$.

Show that if $\zeta = s - \frac{3}{2}t$, $\tau = \frac{1}{2}\varepsilon t$, then $H_\tau = H_{\zeta\zeta}$, and deduce that if $H = \delta\exp[-s^2/l^2]$ at $t = 0$, then

$$H = \delta\left(\frac{t_0}{t_0+t}\right)^{1/2}\exp\left[\frac{-(s-\frac{3}{2}t)^2}{2\varepsilon(t_0+t)}\right]$$

for $t > 0$, where $t_0 = \dfrac{l^2}{2\varepsilon}$. (A typical hydrograph described by this function is shown in figure 1. It is asymmetric, but the steep shock-like rise is limited by the linearity of the model.)

10. The St. Venant equations, assuming Manning's roughness law, zero mass input, and a triangular river cross section, can be written in the dimensionless form

$$A_t + (Au)_s = 0,$$

$$F^2(u_t + uu_s) = 1 - \frac{u^2}{A^{2/3}} - \frac{A_s}{2A^{1/2}}.$$

Show in detail that small disturbances to the steady state $A = u = 1$ can propagate up and down stream if $F < F_1$, but can only propagate downstream if $F > F_1$, and that they are unstable if $F > F_2$. What are the values of F_1 and F_2?

11. *The hydraulic jump*

Using the dimensionless form of the mass and momentum equations (for a canal), show that discontinuities (shocks) in the channel depth travel at a (dimensionless) speed V given by

$$V = \frac{[Au]_-^+}{[A]_-^+} = \frac{[F^2Au^2 + \frac{1}{2}A^2]_-^+}{[F^2Au]_-^+},$$

where $\pm$ refer to the values on either side of the jump, and F is the Froude number. Show that a stationary jump at $s = 0$ is possible (this can be seen when a tap is run into a flat basin) if $Au = Q$ in $s > 0$ and $s < 0$, and

$$\left[\frac{F^2Q^2}{A} + \frac{A^2}{2}\right]_-^+ = 0.$$

Deduce that for prescribed Q and A_-, a unique choice of $A_+ \neq A_-$ is possible. Show also that the locally defined Froude number is

$$Fr = \frac{FQ}{A^{3/2}},$$

and deduce that the hydraulic jump connects a region of *supercritical* ($Fr > 1$) flow to a *subcritical* ($Fr < 1$) one. (In practice, $A_- < A_+$ if $Q > 0$; if $A_- > A_+$, the discontinuity cannot be maintained.)

12. Just as the straightforward St. Venant model is unable to predict the occurrence of transverse dunes, it is also apparently unable to produce lateral bars; at least, this is suggested by the following example.

    Show that a two-dimensional form of the St. Venant equations describing flow in a stream of constant width, which allows for downslope sediment transport, can be written in the dimensionless form

$$s_t + \boldsymbol{\nabla}.\,\mathbf{q} = 0,$$

$$\mathbf{q} = \frac{q(\tau_e)}{\tau_e}\boldsymbol{\tau}_e,$$

$$\boldsymbol{\tau}_e = |\mathbf{u}|\mathbf{u} - \beta\boldsymbol{\nabla}s,$$

$$\varepsilon h_t + \boldsymbol{\nabla}.\,(h\mathbf{u}) = 0,$$

$$F^2\left[\varepsilon\mathbf{u}_t + (\mathbf{u}\,.\boldsymbol{\nabla})\mathbf{u}\right] = -\boldsymbol{\nabla}\eta + \delta\left(\mathbf{i} - \frac{|\mathbf{u}|\mathbf{u}}{h}\right),$$

$$h = \eta - s.$$

    Assume that $\beta \sim O(1)$, $F \sim O(1)$, $\delta \ll 1$, and $\varepsilon \ll 1$. Suppose also that the cross stream width $y \sim \nu \ll 1$. Show that it is appropriate to rescale the transverse velocity v (i.e., $\mathbf{u} = (u, v)$) as $v \sim \nu$, and then also $s \sim \nu^2$ and $t \sim \nu^2$. Assuming that $\varepsilon \ll \nu^2$ and that $q = \tau_e^{3/2}$, show that a consistent approximate rescaled model is

$$\frac{\partial s}{\partial t} + \frac{\partial(u^3)}{\partial x} + \frac{\partial(u^2v)}{\partial y} = \beta\frac{\partial}{\partial y}\left(u\frac{\partial s}{\partial y}\right),$$

$$\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0,$$

$$F^2(uu_x + vu_y) + \eta_x = 0,$$

and that $\eta \approx \eta(x,t)$, $h \approx \eta$. Deduce that s satisfies the equation

$$\frac{\partial s}{\partial t} = \left(\frac{2u}{F^2} + \frac{u^3}{h}\right) h_x + \beta \frac{\partial}{\partial y}\left(u \frac{\partial s}{\partial y}\right).$$

For small perturbations to the uniform state $h = u = 1$, $s = 0$, show that $F^2 uu_x + h_x \approx 0$ in a linearised approximation, and deduce that $u \approx u(x,t)$. Show that then s relaxes to a steady state, and by considering suitable boundary conditions at the stream margins, show that in fact $h_x = 0$, and hence the uniform state is stable.

13. In a model of dune formation, the sediment concentration c and bed height s are modelled by the equations

$$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(hcu) = \rho_s(v_E - v_D),$$

$$(1-n)\frac{\partial s}{\partial t} = -(v_E - v_D),$$

where h is fluid depth, u is mean fluid velocity, ρ_s is sediment density, n is bed porosity, and v_E and v_D are erosion and deposition rates. Parker (J. Fluid Mech. **89**, p. 109 (1978)) suggests the following expressions for the erosion and deposition rates in a stream:

$$v_E = \frac{\beta u_*^3}{v_s^2}, \quad v_D = \frac{\gamma v_s^2 c}{\rho_s u_*},$$

where c is the sediment concentration (mass per unit volume), v_s is the settling velocity, u_* is the 'friction' velocity $(\tau/\rho_w)^{1/2}$, and β and γ are constants ($\approx$ 0.007 and 13, respectively).

Consider the case where the surface $\eta = h + s$ is flat, and $\eta = h_0$ is constant. Assuming $\tau = f\rho_w u^2$ and $uh = q$ is constant, find appropriate scales for x, t and c if $h, \eta, s \sim h_0$ and q is the fluid flux per unit width. Hence derive the dimensionless model for slow flow

$$\varepsilon\frac{\partial}{\partial t}(hc) + \frac{\partial c}{\partial x} = \frac{1}{h^3} - ch = -\frac{\partial s}{\partial t},$$

where

$$\varepsilon = \frac{c_0}{\rho_s(1-n)}.$$

By analysing the stability of the basic state $h = c = \eta = 1$, show that, for ε small, the steady state is stable. What direction do the waves propagate?

More generally, derive a stability criterion in case (i) (when ε is small) if $v_E = E(h)$, $v_D = cV(h)$. How is the result affected if ε is *not* small?

14. The depth of a glacier satisfies the equation

$$H_t + \frac{\partial}{\partial x}\left[(1-\mu H_x)^n \frac{H^{n+2}}{n+2}\right] = B'(x),$$

where $\mu \ll 1$. Suppose first that $\mu \ll 1$, so that the diffusion term can be neglected. Write down the characteristic solution for an arbitrary initial depth profile. What is the criterion on the initial profile which determines whether shocks will form?

Now suppose $B = \dfrac{1}{(n+2)}$ is constant, so that a uniform steady state is possible. Describe the evolution of a perturbation consisting of a uniform increase in depth between $x = 0$ and $x = 1$, and draw the characteristic diagram.

Shock structure. By allowing $\mu \neq 0$, the shock structure is described by the local rescaling $x = x_s(t) + \mu X$. Derive the resulting leading order equation for H, and find a first integral satisfying the boundary conditions $H \to H_\pm$ as $X \to \pm\infty$, where $H_- > H_+$ are the values behind and ahead of the shock. Deduce that the shock speed is

$$\dot{x}_s = \frac{[H^{n+2}]_-^+}{[H]_-^+},$$

and that $\phi = H/H_+$ satisfies the equation

$$\phi_\xi = -[g(\phi)^{1/n} - 1],$$

where $\xi = X/H_+$, $\phi \to r$ as $\xi \to -\infty$, $\phi \to 1$ as $\xi \to \infty$, and

$$g(\phi) = \frac{(r^{n+2}-1)(\phi-1) + (r-1)}{(r-1)\phi^{n+2}},$$

with $r = H_-/H_+ > 1$. Show that $g(1) = g(r) = 1$, and that $g(\phi) > 1$ for $1 < \phi < r$, and deduce that a monotonic shock structure solution joining H_- to H_+ does indeed exist.

Suppose that $\delta = \Delta H/H_+$ is small, where $\Delta H = H_- - H_+$. By putting $r = 1+\delta$ and $\phi = 1 + \delta\Phi$, show that

$$g = 1 + \frac{\delta^2(n+1)(n+2)}{2}\Phi(1-\Phi) + \ldots,$$

and deduce that

$$\Phi_\Xi \approx -\Phi(1-\Phi),$$

where

$$\Xi = \frac{\delta(n+1)(n+2)}{2n}\xi.$$

Deduce that the width of the shock structure is of dimensionless order

$$x - x_s \sim \frac{2n\mu H_+}{\delta(n+1)(n+2)},$$

or dimensionally

$$\frac{2n}{(n+1)(n+2)}\frac{d_+^2}{\Delta d\,\tan\alpha},$$

and that for a glacier of depth 100 m, slope ($\tan\alpha$) 0.1, with $n = 3$, a wave of height 10 m has a shock structure of width 3 km.

15. Write down suitable equations describing the flow of a two-dimensional ice sheet over a flat bed assuming no slip at the bed, and show, using the shallow ice approximation, that the surface elevation $z = s(x,t)$ satisfies a diffusion equation of the form

$$s_t = [D_I s_x]_x + a,$$

where you should give an expression for D_I. What are suitable boundary conditions describing a grounded ice sheet? What would you expect the corresponding model of a three-dimensional ice sheet to be? Assuming a two-dimensional model in which the ice summit is at $x = 0$ and

$$a = a_0\left(1 - \frac{2x}{l}\right),$$

make the model suitably dimensionless, and write down a steady state solution for the ice depth in the form of an integral, and give an expression for the maximum ice depth.

Suppose now a layer of dust of depth h lies on the surface, which satisfies the transport equation

$$h_t + (uh)_x = D - E;$$

give a physical interpretation of the terms in this equation.

Suppose that $a(h)$ is a decreasing function of h (dust-albedo feedback), D is constant, $E(h,u)$ is an increasing function of both its arguments, and the *katabatic* wind speed u is given by

$$u = -Ks_x.$$

By assuming perturbations to a locally uniform state in which

$$s = s_0 - Sx, \quad h = h_0,$$

where s_0, S and h_0 are constants, show that instabilities of the ice surface can occur in the form of growing travelling waves, if h_0 (and thus D) is sufficiently large. Give an approximate criterion for the instability to occur, and show that the waves move downslope.

[*Hint: it may be useful to assume that of the two possible modes, one is fast (and stable), while the possibly unstable one is slow.*]