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# Stability of oblique liquid curtains with surface tension

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## AFFILIATIONS

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## ABSTRACT

Oblique (non-vertical) liquid curtains are examined under the assumption that the Froude number is large. As shown previously [E. S. Benilov, "Oblique liquid curtains with a large Froude number," J. Fluid Mech. **861**, 328 (2019)], their structure depends on the Weber number: if $We < 1$ (strong surface tension), the Navier–Stokes equations admit asymptotic solutions describing curtains bending upward, i.e., against gravity. In the present paper, it is shown that such curtains are unstable with respect to small perturbations of the flow parameters at the outlet: they give rise to a disturbance traveling downstream and becoming singular near the curtain's terminal point (where the liquid runs out of the initial supply of kinetic energy). It is argued that, since the instability is spatially localized, the curtain can be stabilized by a properly positioned collection nozzle. All curtains with $We > 1$ bend downward and are shown to be stable.



## I. INTRODUCTION

Liquid curtains have been part of classical hydrodynamics for more than 60 years since the seminal work of G. I. Taylor (described in the appendix of Ref. 1). They arise when the liquid is ejected through a long narrow slot (outlet); if it is ejected obliquely, the curtain's trajectory is curved by gravity.

The present work is concerned with oblique curtains bending upward. Such solutions were first obtained via a semi-phenomenological model[2] and then rediscovered via straightforward asymptotic analysis of the Navier–Stokes equations.[3] This counter-intuitive result was criticized by Ref. 4, who noted that the asymptotic equations used to derive it are hyperbolic, and one of the characteristics corresponds to waves propagating upstream. According to Ref. 4, this circumstance necessitates that the boundary condition prescribing the ejection angle be replaced with the requirement that the curtains fall vertically down. This argument was disputed in Ref. 5, who pointed out that the change of the exit condition could be justified physically only if a mechanism existed forcing the curtain to make an abrupt turn after exiting the outlet—but none does. To clarify the issue, Ref. 5 derived a more accurate, albeit less general, asymptotic model based on non-hyperbolic equations (to which the criticisms of Ref. 4 definitely do not apply), and this model was shown to also admit solutions describing upward-bending curtains. Finally, Ref. 5 argued that the question of observability of upward-bending curtains can only be resolved via a stability study.

The present paper delivers just that. Since its goal is physical understanding rather than detailed modeling, the simplest setting with waves propagating upstream is considered: non-sheared curtains with a large Froude number.

In Sec. II, it is shown that the temporal stability analysis (where the frequencies of harmonic disturbances are sought as the eigenvalues of the linearized problem) yields no solutions—neither stable nor unstable. Such a result often implies stability—e.g., for the Couette and Poiseuille flows in an inviscid fluid.[6,7] In the present case, however, it does not: as shown in Sec. III, the flow is convectively unstable with respect to a vibration of the outlet and/or perturbation of the curtain's ejection velocity and/or angle. It should be emphasized that convective instability and temporal stability rarely, if ever, arise in the same setting (e.g., Ref. 8).

It is also shown in Sec. III that all downward-bending curtains are stable.

## II. FORMULATION OF THE PROBLEM

### A. Governing equations

Consider a sheet of incompressible liquid (density $\rho$ and surface tension $\sigma$) ejected from an infinitely long horizontal outlet—see Fig. 1. Let the flow be two-dimensional and depend on a horizontal variable $x$ and vertical coordinate $z$. The shape of the sheet can be conveniently described by a parametric representation of its centerline,

$$x = x(l,t), \quad z = z(l,t),$$

where the parameter $l$ is the centerline's arc length measured from the midpoint of the outlet, and $t$ is the time. It can be shown that the coordinates of the centerline satisfy

$$\frac{\partial x}{\partial l} = \cos\alpha, \quad \frac{\partial z}{\partial l} = \sin\alpha, \tag{1}$$

where $\alpha(l,t)$ is the angle between the sliding tangent to the centerline and the horizontal.

To nondimensionalize the problem, introduce the half-width $H$ of the outlet and use it as a scale for the curtain's half-thickness $h$. The natural scale for the streamwise velocity in this problem is

$$U = \left(\frac{\sigma}{\rho H}\right)^{1/2}.$$

The centerline's coordinates $(x, z)$ and arc length $l$ will be nondimensionalized on

$$L = \frac{\sigma}{\rho g H},$$

where $g$ is the acceleration due to gravity. The time variable $t$ will be nondimensionalized on $U/L$.

Let the effect of gravity be weaker than inertia, so that the Froude number is large,

$$\frac{U^2}{gH} \gg 1. \tag{2}$$

Since gravity bends the curtain, whereas inertia keeps it straight, this assumption makes the curtain's curvature small and, thus, enables one to take advantage of the slender-curtain approximation.

Assumption (2) alone is sufficient for solving the problem at hand—but to make it simpler, assume also that the fluid is highly viscous,

$$\frac{gH^2}{\nu U} \ll 1, \tag{3}$$

where $\nu$ is the kinematic viscosity. In this case, the streamwise velocity $u$ can be assumed to be non-sheared, i.e., independent of the cross-stream coordinate (which will not be introduced). Thus, $u$ depends only on $l$ and $t$—which dramatically simplifies the calculations involved and lets one concentrate on the qualitative results.

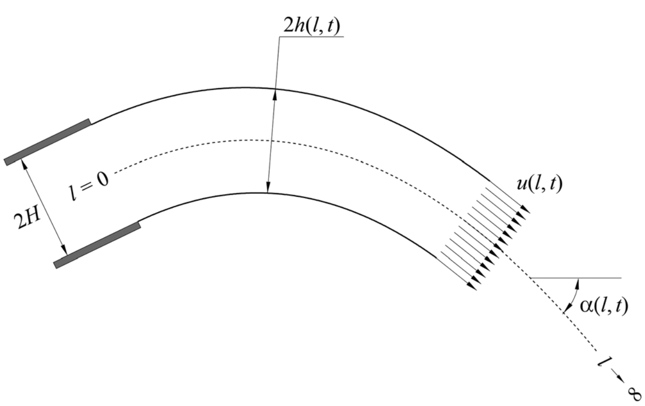


FIG. 1. Setting.

Assumptions (2) and (3) were used in Ref. 3 to derive the following set of equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial l} = -\sin\alpha - \frac{\partial^2 x}{\partial t^2}\cos\alpha - \frac{\partial^2 z}{\partial t^2}\sin\alpha, \tag{4}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial l} = 0, \tag{5}$$

$$2u\frac{\partial \alpha}{\partial t} + \left(u^2 - \frac{1}{h}\right)\frac{\partial \alpha}{\partial l} = -\cos\alpha + \frac{\partial^2 x}{\partial t^2}\sin\alpha - \frac{\partial^2 z}{\partial t^2}\cos\alpha. \tag{6}$$

To better understand the physics behind Eqs. (4)–(6), one needs to identify the physical meaning of the terms in the equations derived. While doing so, keep in mind that the coordinates system $(l, \tau)$ depends on $t$ and, thus, is non-inertial.

- The terms involving $\partial^2 x/\partial^2 t$ and $\partial^2 z/\partial^2 t$ in Eqs. (4) and (6) describe the force of inertia due to the curtain's local acceleration.
- The first term in Eq. (6) describes the Coriolis force, with $\partial\alpha/\partial t$ being the local angular velocity of the curtain's rotation.
- The second term in Eq. (6) describes the centripetal acceleration of liquid particles and the acceleration due to the capillary pressure, with $\partial\alpha/\partial l$ being the curtain's local curvature.
- The terms involving $\sin\alpha$ in Eq. (4) and $\cos\alpha$ in Eq. (6) describe gravity.

It remains to fix the boundary conditions,

$$(x, z, u, h, \alpha) = (x_0, z_0, u_0, h_0, \alpha_0) \quad \text{at} \quad l = 0. \tag{7}$$

Since this paper is concerned with stability of curtains with respect to vibration of the outlet and perturbation of the ejection parameters, one should let $x_0$, $z_0$, $u_0$, $h_0$, and $\alpha_0$ be functions of $t$.

Given a suitable initial condition, set (1)–(7) determine the unknowns $(x, z, u, h, \alpha)$ as functions of $(l, t)$.

## B. Steady curtains

Steady curtains arise when the coefficients $(x_0, z_0, u_0, h_0, \alpha_0)$ in the exit conditions do not depend on $t$. Without loss of generality, one can then assume that the outlet is located at $(x, z) = (0, 0)$ and its nondimensional half-thickness equals unity, so that Eq. (7) becomes

$$(x, z, u, h, \alpha) = (0, 0, \bar{u}_0, 1, \bar{\alpha}_0) \quad \text{at} \quad l = 0,$$

where $\bar{u}_0$ and $\bar{\alpha}_0$ are constants. Let the unknowns be also time-independent,

$$x = \bar{x}(l), \quad z = \bar{z}(l), \quad u = \bar{u}(l), \quad h = \bar{h}(l), \quad \alpha = \bar{\alpha}(l).$$

Then, (1)–(6) become ordinary differential equations (ODEs); the ODE for $\bar{x}$ decouples from the rest of the set, whereas $\bar{z}$, $\bar{h}$, and $\bar{\alpha}$ are related to $\bar{u}$,

$$\bar{z} = \frac{\bar{u}_0^2 - \bar{u}^2}{2}, \quad \bar{h} = \frac{\bar{u}_0}{\bar{u}}, \quad \cos\bar{\alpha} = \frac{(\bar{u}_0^2 - 1)\cos\bar{\alpha}_0}{\bar{u}\bar{u}_0 - 1}. \tag{8}$$

It is shown in Ref. 3 that if $\bar{u}_0 < 1$, the curtain bends upward, whereas curtains with $\bar{u}_0 > 1$ bend downward (the value of the ejection angle

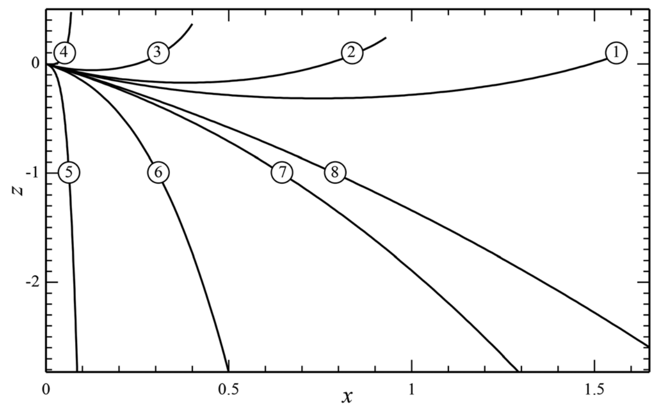


FIG. 2. Trajectories of steady curtains with $\bar{\alpha}_0 = -\pi/4$. Curves (1)–(8) correspond to $\bar{u}_0 = 0.5,\ 0.7,\ 0.9,\ 0.99,\ 1.01,\ 1.1,\ 1.5,$ and 2, respectively.

$\bar{\alpha}_0$ does not affect this pattern). Defining the Weber number as $We = \bar{u}_0^2$, one recovers the criterion separating the two kinds of curtains derived in Refs. 2 and 3.

Typical steady-curtain solutions (computed numerically) are shown in Fig. 2. Observe that upper-bending (UB) curtains have a terminal point—where the streamwise velocity vanishes, whereas its thickness becomes infinite. The slender-curtain approximation fails some distance before this occurs, but the solution in the rest of the domain remains valid. It was conjectured in Ref. 3 that once the curtain reaches its terminal point, the liquid just splashes down.

For downward-bending (DB) curtains, the solution exists and is smooth for all $l \geq 0$.

Also observe that near-critical curtains ($\bar{u}_0 \to 1$)—both UB and DB—can be subdivided into a segment of high curvature, followed by a nearly vertical segment. This tendency can be confirmed via a straightforward asymptotic analysis of the governing equations.

### C. Linear disturbances

The stability of liquid curtains has been examined before—both theoretically[9–14] and experimentally.[1,10,15–17] All these studies, however, were concerned with vertical curtains.

To examine oblique ones, seek a solution of the governing set (1)–(7) in the form

$$x = \bar{x}(l) + \tilde{x}(l,t), \quad z = \bar{z}(l) + \tilde{z}(l,t), \tag{9}$$

$$u = \bar{u}(l) + \tilde{u}(l,t), \quad h = \bar{h}(l) + \tilde{h}(l,t), \tag{10}$$

$$\alpha = \bar{\alpha}(l) + \tilde{\alpha}(l,t), \tag{11}$$

where $\bar{x}(l)$, etc., describe a steady curtain, and $\tilde{x}(l,t)$, etc., represent a small disturbance. The exit conditions should also be perturbed,

$$x_0 = \tilde{x}_0(t), \quad z_0 = \tilde{z}_0(t), \tag{12}$$

$$u_0 = \bar{u}_0 + \tilde{u}_0(t), \quad h_0 = \bar{h}_0 + \tilde{h}_0(t), \quad \alpha_0 = \bar{\alpha}_0 + \tilde{\alpha}_0(t). \tag{13}$$

Substituting Eqs. (9)–(13) in Eqs. (1)–(7) and linearizing the latter, one obtains

$$\frac{\partial \tilde{x}}{\partial l} = -\tilde{\alpha}\sin\bar{\alpha}, \quad \frac{\partial \tilde{z}}{\partial l} = \tilde{\alpha}\cos\bar{\alpha}, \tag{14}$$

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial(\bar{u}\tilde{u})}{\partial l} = -\tilde{\alpha}\cos\bar{\alpha} - \frac{\partial^2\tilde{x}}{\partial t^2}\cos\bar{\alpha} - \frac{\partial^2\tilde{z}}{\partial t^2}\sin\bar{\alpha}, \tag{15}$$

$$\frac{\partial \tilde{h}}{\partial t} + \frac{\partial}{\partial l}\left(\bar{u}\tilde{h} + \tilde{u}\bar{h}\right) = 0, \tag{16}$$

$$2\bar{u}\frac{\partial\tilde{\alpha}}{\partial t} + \left(\bar{u}^2 - \frac{1}{\bar{h}}\right)\frac{\partial\tilde{\alpha}}{\partial l} + \left(2\bar{u}\tilde{u} + \frac{\tilde{h}}{\bar{h}^2}\right)\frac{\partial\bar{\alpha}}{\partial l}$$
$$= \tilde{\alpha}\sin\bar{\alpha} + \frac{\partial^2\tilde{x}}{\partial t^2}\sin\bar{\alpha} - \frac{\partial^2\tilde{z}}{\partial t^2}\cos\bar{\alpha}, \tag{17}$$

$$\tilde{x} = \tilde{x}_0, \quad \tilde{z} = \tilde{z}_0 \quad \text{at} \quad l = 0, \tag{18}$$

$$\tilde{u} = \tilde{u}_0, \quad \tilde{h} = \tilde{h}_0, \quad \tilde{\alpha} = \tilde{\alpha}_0 \quad \text{at} \quad l = 0. \tag{19}$$

This boundary-value problem can be rewritten in terms of the Fourier transforms of the tilded variables—or simply by assuming that they oscillate harmonically with a frequency $\omega$, i.e.,

$$\tilde{x} = \hat{x}(l)e^{-i\omega t}, \quad \tilde{z} = \hat{z}(l)e^{-i\omega t},$$
$$\tilde{u} = \hat{u}(l)e^{-i\omega t}, \quad h = \hat{h}(l)e^{-i\omega t}, \quad \alpha = \hat{\alpha}(l)e^{-i\omega t},$$
$$\tilde{x}_0 = \hat{x}_0e^{-i\omega t}, \quad \tilde{z}_0 = z_0e^{-i\omega t},$$
$$\tilde{u}_0 = \hat{u}_0e^{-i\omega t}, \quad \tilde{h}_0 = \hat{h}_0e^{-i\omega t}, \quad \tilde{\alpha}_0 = \hat{\alpha}_0e^{-i\omega t}.$$

As a result, (14)–(19) become

$$\frac{d\hat{x}}{dl} = -\hat{\alpha}\sin\bar{\alpha}, \quad \frac{d\hat{z}}{dl} = \hat{\alpha}\cos\bar{\alpha}, \tag{20}$$

$$-i\omega\hat{u} + \frac{d(\bar{u}\hat{u})}{dl} = -\hat{\alpha}\cos\bar{\alpha} + \omega^2(\hat{x}\cos\bar{\alpha} + \hat{z}\sin\bar{\alpha}), \tag{21}$$

$$-i\omega\hat{h} + \frac{d}{dl}\left(\bar{u}\hat{h} + \hat{u}\bar{h}\right) = 0, \tag{22}$$

$$-2i\omega\bar{u}\hat{\alpha} + \left(\bar{u}^2 - \frac{1}{\bar{h}}\right)\frac{d\hat{\alpha}}{dl} + \left(2\bar{u}\hat{u} + \frac{\hat{h}}{\bar{h}^2}\right)\frac{d\bar{\alpha}}{dl}$$
$$= \hat{\alpha}\sin\bar{\alpha} - \omega^2(\hat{x}\sin\bar{\alpha} - \hat{z}\cos\bar{\alpha}), \tag{23}$$

$$\left(\hat{x}, \hat{z}, \hat{u}, \hat{h}, \hat{\alpha}\right) = \left(\hat{x}_0, \hat{z}_0, \hat{u}_0, \hat{h}_0, \hat{\alpha}_0\right) \quad \text{at} \quad l = 0. \tag{24}$$

### D. How one should define stability in this problem?

Stability of curtains can be examined via boundary-value problem (20)–(24) in two different ways.

(i) One can eliminate the outlet perturbations by setting

$$\hat{x}_0 = \hat{z}_0 = \hat{u}_0 = \hat{h}_0 = \hat{\alpha}_0 = 0, \tag{25}$$

and solve (20)–(25) as an eigenvalue problem for $\omega$. The curtain is unstable if and only if there exists an eigenvalue with $\operatorname{Im}\omega > 0$. Such solutions describe self-amplifying disturbances developing from an initial condition, no matter how small.

(ii) Alternatively, one can assume that $\omega$ is given—physically, this implies that the exit parameters oscillate with a given frequency. The stability of DB and UB curtains in this approach should be defined differently (because the latter exist on a finite domain in $l$).

(a) A DB curtain is stable if and only if the solution of Eqs. (20)–(24) for all $\omega$ and $(\hat{x}_0, \hat{z}_0, \hat{u}_0, \hat{h}_0, \hat{\alpha}_0)$ is such that

$$\lim_{l\to\infty}\left(\left|\frac{\hat{x}}{\bar{x}}\right| + \left|\frac{\hat{z}}{\bar{z}}\right| + \left|\frac{\hat{u}}{\bar{u}}\right| + \left|\frac{\hat{h}}{\bar{h}}\right| + \left|\frac{\hat{\alpha}}{\bar{\alpha}}\right|\right) < \infty, \tag{26}$$

i.e., the disturbance does not outgrow the base flow in the downstream direction.

(b) A UB curtain is stable if and only if the solution of Eqs. (20)–(24) for all $\omega$ and $(\hat{x}_0, \hat{z}_0, \hat{u}_0, \hat{h}_0, \hat{\alpha}_0)$ remains finite (does not involve a singularity).

Approaches (i) and (ii) are often called "temporal" and "spatial," respectively (e.g., Ref. 8) One can also distinguish two kinds of the latter instability: the absolute instability (when the disturbance at a given $l$ grows with $t$) and the convective instability (when disturbances are steady, but grow downstream).

The temporal approach is used much more often than the spatial one—but, in the present case, the former yields the trivial solution only. This is evident from the fact that ODEs (20)–(23) are homogeneous, and all five boundary conditions (24) are fixed at the same end point. Thus, the solution can be obtained by "shooting" from this end point, and since the initial values of all of the unknowns are zero [due to Eq. (25)], so is the whole solution.

Approach (ii) is examined in Sec. III.

## III. STABILITY OF CURTAINS WITH RESPECT TO PERTURBATIONS OF THE EXIT PARAMETERS

### A. Downward-bending curtains ($\bar{u}_0 > 1$)

To prove that a DB curtain is stable, it is sufficient to show that all five linearly independent solutions of equations (20)–(23) are bounded as $l \to \infty$. This will be done by changing the independent variable $l \to \xi$, where $\xi = \bar{u}\bar{u}_0 - 1$, and letting

$$\omega = \bar{u}_0\, \omega_{\text{new}}, \quad u = \frac{u_{\text{new}}}{\bar{u}_0}.$$

Omitting the subscript $_{\text{new}}$, one can rewrite (20)–(23) in the form

$$\bar{u}_0^2 \frac{d\hat{x}}{d\xi} = (\xi+1)\hat{\alpha}, \quad \bar{u}_0^2 \frac{d\hat{z}}{d\xi} = \frac{C(\xi+1)}{\sqrt{\xi^2 - C^2}}\hat{\alpha}, \tag{27}$$

$$\frac{\sqrt{\xi^2 - C^2}}{(\xi+1)\xi}\frac{d}{d\xi}[(\xi+1)\hat{u}] - i\omega\hat{u} = -\frac{C}{\xi}\hat{\alpha} + \omega^2\bar{u}_0^2\left(\frac{C}{\xi}\hat{x} - \frac{\sqrt{\xi^2 - C^2}}{\xi}\hat{z}\right), \tag{28}$$

$$\frac{\sqrt{\xi^2 - C^2}}{(\xi+1)\xi}\frac{d}{d\xi}\left[(\xi+1)\hat{h} + \frac{\bar{u}_0^2}{\xi+1}\hat{u}\right] - i\omega\hat{h} = 0, \tag{29}$$

$$\sqrt{\xi^2 - C^2}\frac{d\hat{\alpha}}{d\xi} + \left[\frac{\sqrt{\xi^2 - C^2}}{\xi} - 2i\omega(\xi+1)\right]\hat{\alpha} - \frac{C}{\xi^2}\left(2\hat{u} + \frac{\xi+1}{\bar{u}_0^2}\hat{h}\right) = \omega^2\bar{u}_0^2\left(\frac{\sqrt{\xi^2 - C^2}}{\xi}\hat{x} + \frac{C}{\xi}\hat{z}\right). \tag{30}$$

where

$$C = \left(\bar{u}_0^2 - 1\right)\cos\bar{\alpha}_0.$$

Note that the coefficients of equations (27)–(30) are explicit [unlike those of Eqs. (20)–(23), which are determined by the steady-curtain ODEs].

Note that if the streamwise velocity $\bar{u}$ is a non-monotonic function of $l$, the change $l \to \xi$ is singular at the point where $\bar{u}(l)$ has an extremum. This circumstance does not pose a problem; however, as criterion (26) implies that the stability properties of a DB curtain depend on the *large-distance* asymptotics of the solution, i.e., in the limit $l \to \infty$ (corresponding to $\xi \to \infty$). In this case, the curtain is falling almost vertically, and $\bar{u}(l)$ is indeed monotonic (increasing). In principle, $\bar{u}(l)$ of a DB curtain can be monotonic *globally*, but only if $\bar{\alpha}_0 \le 0$ (otherwise $\bar{u}(l)$ first decreases and then starts to increase).

Denote the general solution of the (fifth-order) linear set (27)–(30) by

$$\psi = \left[\hat{x}(\xi),\, \hat{z}(\xi),\, \hat{u}(\xi),\, \hat{h}(\xi),\, \hat{\alpha}(\xi)\right],$$

and represent it in the form

$$\psi = \sum_{n=1}^{5} c_n\psi_n,$$

where $c_n$ are arbitrary constants and $\psi_n$ are linearly independent solutions. $\psi_n$ can be fixed by their asymptotic behaviors at infinity; these behaviors had to be guessed—but, once they are, one can simply verify the result via straightforward calculations.

The following asymptotic solutions of equations (27)–(30) have been found for the limit $\xi \to \infty$:

$$\psi_1 \sim \left[-C\xi^{-1},\, 1,\, -i\omega\bar{u}_0^2,\, 2\bar{u}_0^4\xi^{-3},\, \bar{u}_0^2 C\xi^{-3}\right],$$

$$\psi_{2,3} \sim \left[1,\, C\xi^{-1}, -2\bar{u}_0^2 C\xi^{-2}, \pm 2\bar{u}_0^4 C\xi^{-7/2}, i\omega\bar{u}_0^2\xi^{-1}\right] \times\xi^{1/4+i\omega}e^{i\omega(\xi\pm 2\xi^{1/2})},$$

$$\psi_4 \sim \left[\frac{C}{\omega^2\bar{u}_0^4}\xi^{-1}, \frac{C^2}{\omega^2\bar{u}_0^4}\xi^{-2}, -\frac{(i\omega+4)C^2}{3\omega^2\bar{u}_0^2}\xi^{-4}, \xi^{-1}, -\frac{C}{i\omega\bar{u}_0^2}\xi^{-2}\right]e^{i\omega\xi},$$

$$\psi_5 \sim \left[\frac{2(1+i\omega)C}{\omega^2\bar{u}_0^2}\xi^{-2}, \frac{2(1+i\omega)C^2}{\omega^2\bar{u}_0^2}\xi^{-3}, \xi^{-1}, 2i\omega\bar{u}_0^2\xi^{-2}, -\frac{2(1+i\omega)C}{i\omega}\xi^{-3}\right]e^{i\omega\xi}.$$

One with prior experience of working with curtains or jets can identify the physical meanings of these solutions: $\psi_1$ describes oscillations of the curtain as a whole, $\psi_2$ and $\psi_3$ describe sinuous capillary waves (both propagate downstream), $\psi_4$ describes varicose disturbances, and $\psi_5$ is a stretching mode.

Recall that the stability criterion (26) involves also the large-distance asymptotics of the steady curtain—for which (8) and (1) yield

$$\bar{x} \sim \frac{C}{\bar{u}_0^2}\xi, \quad \bar{z} \sim -\frac{1}{2\bar{u}_0^2}\xi^2, \quad \bar{u} \sim \frac{1}{\bar{u}_0}\xi, \quad \bar{h} \sim \bar{u}_0\xi^{-1},$$
$$\bar{\alpha} \to -\frac{\pi}{2} \quad \text{as} \quad \xi \to \infty.$$

Evidently, criterion (26) holds for any linear combination of solutions $\psi_n$.

### B. Upward-bending curtains ($\bar{u}_0 < 1$)

To prove that UB curtains are unstable, it is sufficient to find a single singular solution of set (27)–(30), but it turns out that a three-parameter family of such exists. Keeping in mind that the terminal point corresponds to $\xi \to -1$, one can guess the asymptotics of these solutions,

$$\psi \sim \left[A_1,\, A_2,\, A_3(\xi+1)^{-1},\, A_3\bar{u}_0^2(\xi+1)^{-3},\, -\frac{A_3 C}{\sqrt{1-C^2}}(\xi+1)^{-1}\right],$$

where $A_1$, $A_2$, and $A_3$ are arbitrary constants. Observe that, even though the perturbations of the curtain's coordinates remain finite, those of the streamwise velocity, width, and direction tend to infinity. Note also that the disturbance $\hat{h}$ of the curtain's width grows faster than the width $\bar{h}$ itself, which can be verified by deducing from Eq. (8) that $\bar{h} = \bar{u}_0^2(\xi+1)^{-1}$.

To illustrate the spatial structure of disturbances, boundary-value problem (20)–(24) was solved numerically. It has turned out that little depends on which one of the exit parameters oscillates—or, mathematically speaking, which subset of the constants $(\hat{x}_0, \hat{z}_0, \hat{u}_0, \hat{h}_0, \hat{\alpha}_0)$ in boundary conditions (24) are chosen to be non-zero. The dependence on the frequency $\omega$ and ejection angle $\bar{\alpha}_0$ is also weak, so the ejection velocity $\bar{u}_0$ is the only important parameter.

The typical structure of the disturbance is shown in Fig. 3: one can see that the growth is confined to a narrow neighborhood of the curtain's terminal point. It can be shown analytically that, in the limit $\bar{u}_0 \to 1^-$ (near-critical UB curtains), this tendency becomes even stronger.

## IV. CONCLUDING REMARKS

Four points remain to be discussed.

First, the localized nature of instability of upper-bending curtains suggests that they can be stabilized by "collecting" the whole curtain via a collection nozzle positioned just below the terminal point. This way, the unstable disturbances would never grow to a significant level, but it remains to be seen if this can be done in a real experiment.

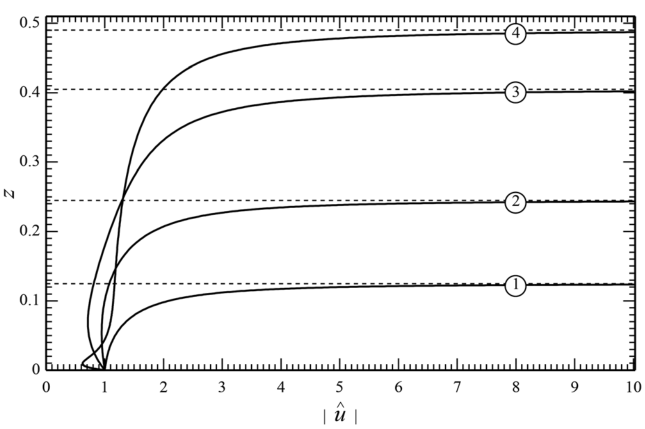


FIG. 3. The vertical structure of the disturbance as described by boundary-value problem (16)–(20) with $\omega = 1$, $\bar{\alpha}_0 = 0$, $\hat{u}_0 = 1$, and $\hat{x}_0 = \hat{z}_0 = \hat{h}_0 = \hat{\alpha}_0 = 0$. Curves (1)–(4) correspond to $\bar{u}_0 = 0.5,\ 0.7,\ 0.9,$ and $0.99$, respectively. The dotted line shows the maximum height of the curtain (the amplitude of the disturbance tends to infinity when approaching this point).

Second, the setting examined in this paper is not the first example where the analysis of temporal instability is not representative of the stability properties of the flow. The first example was reported in Refs. 18 and 19 for a liquid film in a rotating horizontal cylinder. It was shown that a complete set of stable temporal modes exists—and yet the flow is unstable with respect to non-harmonic perturbations developing a singularity in a finite time. This situation is, however, different from the one examined in this paper, where the temporal analysis yields neither stable nor unstable solutions.

Third, Ref. 20 found steady solutions describing oblique swirling jets, which also bend against gravity. One can use this example to find out whether an upward-bending flow is destined to be unstable—and if it is, whether or not the instability is spatially localized. It is worth adding here that the only class of "unusually" bending jets examined previously has turned out to be unstable—to the extent that they break down near the nozzle (as shown in Ref. 21 for jets from a rotating nozzle). One should keep in mind, however, that the instability in this case could be caused by capillary effects rather than the jet's curvature.

Fourth, the stability of curtains in this paper has been examined only with respect to along-the-stream perturbations, whereas lateral perturbations (with a wave vector perpendicular to the plane of Fig. 1) have been neglected. For (unstable) upward-bending curtains, lateral perturbations do not change anything—but they can, in principle, destabilize downward-bending curtains. It is highly unlikely, however, that they do, as they correspond to waves propagating along the outlet, not toward it. Even more importantly, there is ample experimental evidence that downward-bending curtains are stable.[16,17,22] At the same time, it is not easy to examine lateral perturbations theoretically, as they require a three-dimensional extension of the governing equations, and there is an intrinsic difficulty with this kind of asymptotic analyses for curtains and jets.[23,24]

## AUTHOR DECLARATIONS

### Conflict of Interest

The author has no conflicts to disclose.

### Author Contributions

**Eugene Benilov:** Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

## DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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