

NONLINEAR WAVES IN WEAKLY DISPERSIVE RANDOM MEDIA

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The propagation of nonlinear waves in random media is an important aspect of nonlinear wave theory and has a long and informative history. This paper describes the basic ideas of the approaches that have been applied. The average-field method, which has been used most extensively in linear problems, is considered. This approach is then shown to be incorrect as far as nonlinear processes are concerned. Finally, a new scheme is proposed average-form the method, which allows consistent evolution equations to be obtained for nonlinear waves in random media.

1. INTRODUCTION

Wave scattering in media with space–time inhomogeneities is among the basic problems in the general theory of wave processes. The problem is reduced in the simplest case to analyzing a linear wave equation with variable random coefficients. Its first (not very rigorous) solution, which is called the average-field method, was proposed by Kaner [1] and Bourret [2] and was substantiated later by Tatarsky [3, 4] with the help of a diagram procedure. Using these results, many authors have been successfully in using the average-field method to solve various problems of plasma physics, radiophysics, acoustics, and oceanology. It is only natural that the problem of extending the results obtained to nonlinear media seems very attractive and important. Such generalizations were made 20 years ago [5–11]. It should be noted, however, that the methods of the diagram procedure have not been extended to nonlinear problems, and additional substantiation was required for use of the average-field method, which is not altogether correct. Moreover, an example of a nonlinear system with fluctuating parameters for which obviously incorrect results were obtained through the average-field method has been suggested [12]. A new impetus came 10 years later [14, 15], when an asymptotically rigorous generalization of the average-field method for a nonlinear one-dimensional wave equation was proposed, which is called the average-form method. This method is now being extended to more-general wave systems [16–18].

The present paper highlights the basic ideas underlying nonlinear generalizations of the average-field method. The approach suggested seems rather effective in solving various physical problems of nonlinear wave field scattering in random media.

2. AVERAGE-FIELD METHOD

Let us consider first the central ideas of nonlinear generalization of the average-field method in the form in which it was introduced 20 years ago [5–11]. The simplest example of the idea can be an operator equation in the form

$$\hat{L}u = \varepsilon \hat{M}u + \mu \hat{Q}u^2, \quad (2.1)$$

where \hat{L} , \hat{M} , and \hat{Q} are determinate linear integrodifferential operators; u is the vector of the functions describing the wave field; μ is a parameter characterizing the nonlinearity; and $\varepsilon(\vec{r}, t)$ is the vector of random functions with a given statistics defining the space–time inhomogeneities of the medium. We choose $\varepsilon(\vec{r}, t)$ such that

$$\langle \varepsilon(\vec{r}, t) \rangle = 0, \quad (2.2)$$

where $\langle \dots \rangle$ is used to denote averaging over the realizations ensemble. Let us write the wave field in the form of average $\langle u \rangle$ and scattered u' fields

$$\mathbf{u} = \langle \mathbf{u} \rangle + \mathbf{u}', \quad \langle \mathbf{u}' \rangle = 0. \quad (2.3)$$

Let us substitute (2.3) into (2.1) and average the equation obtained over the inhomogeneity ensemble taking (2.2) into account

$$\widehat{L}\langle \mathbf{u} \rangle = \langle \varepsilon \widehat{M} \mathbf{u}' \rangle + \mu \widehat{Q}\langle \mathbf{u} \rangle^2 + \mu \widehat{Q}\langle \mathbf{u}'^2 \rangle. \quad (2.4)$$

Equation (2.4) is non-closed with respect to the average field. Another equation will be derived by subtracting (2.4) from (2.1)

$$\widehat{L}\mathbf{u}' = \varepsilon \widehat{M}\langle \mathbf{u} \rangle + (\varepsilon \widehat{M} \mathbf{u}' - \langle \varepsilon \widehat{M} \mathbf{u}' \rangle) + \mu \widehat{Q}(\mathbf{u}'^2 - \langle \mathbf{u}'^2 \rangle) + 2\langle \mathbf{u} \rangle \mathbf{u}'. \quad (2.5)$$

It is hardly easier to obtain solutions of Eqs. (2.4) and (2.5), as compared with the initial equation (2.1), since the former contain a random vector function ε .

An objective of the average-field method is to obtain equations containing determinate functions along (e.g., correlation functions of ε). This procedure is usually realized using some approximations. For example, $\langle \varepsilon^2 \rangle$ can be often assumed to be a small parameter, and this case corresponds to a certain smallness of scattering so that the wave can propagate for a long time while retaining its individual nature. Also, nonlinearity will be assumed

$$\mu \sim \langle \varepsilon^2 \rangle. \quad (2.6)$$

Scattered-field generation is described by the first term in (2.5), and scattered field u' can be expected to be proportional to ε for small ε . In this case, the remaining terms in (2.5) are of the second order (or higher) of smallness and can be omitted in a first approximation. Equation (2.5) is then reduced to the form

$$\widehat{L}\mathbf{u}' = \varepsilon \widehat{M}\langle \mathbf{u} \rangle \quad (2.7)$$

and is readily integrated in quadratures

$$\mathbf{u}' = \widehat{L}^{-1} \varepsilon \widehat{M}\langle \mathbf{u} \rangle. \quad (2.8)$$

(We shall ignore possible technical problems in obtaining the inverse operator \widehat{L}^{-1} , since they have nothing to do with the medium's randomness.)

If we take an unperturbed solution of (2.4) (with zero left side) as $\langle \mathbf{u} \rangle$, all characteristics of a scattered field are easily studied with the help of (2.8); this approximation is called Born's approximation or a single-scattering approximation [3, 4]. Such an approach is usually valid for a small inhomogeneity magnitude and a small volume occupied by the inhomogeneities, and u' remains small as compared with $\langle u \rangle$. One can try, however, to obtain solution of the initial equation that is applicable for large scattering volumes. To this end, by analogy with asymptotic methods, field $\langle u \rangle$ should not be assumed as given or be used in such a way that u' could remain restricted at sufficiently great distances.

The desired equation for $\langle u \rangle$ has already been written (Eq. (2.4)), and the problem can be reduced essentially to calculating the scattered field u' through $\langle u \rangle$ with a certain degree of accuracy in order to make Eq. (2.4) closed. In the first approximation, u' can be naturally related to $\langle u \rangle$ through (2.8) (but with $\langle u \rangle$ not assumed as a given function), and then the equation for the average field takes the form

$$\widehat{L}\langle \mathbf{u} \rangle = \langle \varepsilon \widehat{M} \widehat{L}^{-1} \varepsilon \widehat{M} \rangle \langle \mathbf{u} \rangle + \widehat{Q}\langle \mathbf{u} \rangle^2. \quad (2.9)$$

A closed equation with determinate coefficients results, and its solution can be obtained using regular methods of nonlinear wave theory. We have just described the entire concept of the average-field method, and the approximations themselves were as follows:

Bourret approximation (scattering multiplicity is ignored), under which the terms in the first brackets on the right side of Eq. (2.5) are dropped.

Howe approximation (nonlinearity of the scattered field is ignored), under which the terms of the type $\langle u^2 \rangle$ and $u'x\langle u \rangle$ are dropped in (2.4) and (2.5).

This method is rather attractive, since the smallness of each term seems obvious at the sign level, and, moreover, this method (i.e., Bourret approximation) has been substantiated in linear problems. Difficulties arise, however, in nonlinear problems.

3. NONLINEAR WAVE EQUATION

Let us consider the simplest model of a one-dimensional weakly nonlinear nondispersive medium with small fluctuations of propagation velocity.

$$\frac{\partial^2 u}{\partial t^2} - \left(1 + \mu \varepsilon(x)\right)^2 \frac{\partial^2 u}{\partial x^2} = \mu^2 \frac{\partial^2 u^2}{\partial x^2}. \quad (3.1)$$

Applying the above average-field method to Eq. (3.1) we obtain a closed equation for the average field (for details see [15]):

$$\frac{\partial^2 \langle u \rangle}{\partial t^2} - \frac{\partial^2 \langle u \rangle}{\partial x^2} = \mu^2 \left(2 \int_{-\infty}^{\infty} W(\tau) \frac{\partial^3 \langle u \rangle}{\partial x^2 \partial t} (x - \tau, t - |\tau|) d\tau - 3\sigma^2 \frac{\partial^2 \langle u \rangle}{\partial x^2} + \frac{\partial^2 \langle u \rangle^2}{\partial x^2} \right), \quad (3.2)$$

where $W(\tau) = \langle \varepsilon(x)\varepsilon(x + \tau) \rangle$ is the correlation function of propagation-velocity fluctuations and $\sigma^2 = \langle \varepsilon^2 \rangle = W(0)$ is the variance of fluctuations. The scattered field in this approximation can be written in quadratures in the form

$$u'(x, t) = \mu \int \int_{-\infty}^{\infty} G(x - x_0, t - t_0) 2\varepsilon(x_0) \frac{\partial^2 \langle u \rangle}{\partial x_0^2} (x_0, t_0) dx_0 dt_0. \quad (3.3)$$

Here G is Green's function of a linear wave equation, which is equal to $G(x, t) = 1/2H(t - |x|)$, where H is the Heaviside function.

It seems convenient for a deeper analysis to use a single-wave approximation, which is valid if, for example, the initial conditions are satisfied for a wave travelling in the right-hand direction alone. The solution of Eq. (3.2) will then depend on the variables x and t through the following arguments

$$z = x - t, \quad T = \mu^2 t \quad (3.4)$$

and using standard methods of nonlinear wave theory, which are accurate to within μ^2 , Eq. (3.2) takes the form

$$\begin{aligned} \frac{\partial \langle u \rangle}{\partial T} + \langle u \rangle \frac{\partial \langle u \rangle}{\partial z} - \frac{3\sigma^2}{2} \frac{\partial \langle u \rangle}{\partial z} - \left(\int_0^{\infty} W(\tau) d\tau \right) \frac{\partial^2 \langle u \rangle}{\partial z^2} - \\ - \int_0^{\infty} W(\tau) \frac{\partial^2 \langle u \rangle}{\partial z^2} (z + 2\tau, T) d\tau = 0. \end{aligned} \quad (3.5)$$

This equation coincides with the Burgers equation to within the last term and is reduced to the latter under the condition of small-scale fluctuations. Equation (3.5) is considered the "final product" of the average-field method.

In order to evaluate the applicability of the average-field method it seems convenient to pass to a single-wave approximation in the expression for the scattered field (3.3)

$$u'(x, t) = -\frac{\partial \langle u \rangle}{\partial z} \mu \Theta(x) - \int_0^{\infty} \mu \varepsilon(z + t + \tau) \frac{\partial \langle u \rangle}{\partial z}(z + 2\tau, T) dt, \quad (3.6)$$

where

$$\Theta = \int_{-\infty}^z \varepsilon(\tau) d\tau \quad (3.7)$$

are phase fluctuations. It should be noted that the first term in (3.6) describes the small displacement of the incident wave as a whole (forward scattering). Indeed, the sum of an incident wave and the term responsible for the forward scattering "because" a single term $\langle u(z - \mu\Theta) \rangle$ to within the accuracy mentioned. The mean-square fluctuation of the phase (Θ^2) is easily seen to be unrestricted with an increase of x

$$\langle \Theta^2 \rangle \rightarrow 2x \int_0^{-\infty} W(\tau) dt. \quad (3.8)$$

It is evident that $\langle \mu'^2 \rangle$ is infinitely large along with $\langle \Theta^2 \rangle$, i.e., this term is dropped when passing from exact equations (2.4) and (2.5) to the approximate equation for an average field (2.9) to Eq. (3.2) in this case. Therefore, the average-field method is found to be incorrect in nonlinear problems. The above does not refer to linear media, where the term of type $\langle u'^2 \rangle$ is not observed at all and the average-field method is reliable enough.

4. THE AVERAGE FORM METHOD

It becomes evident from the above that the inapplicability of the average field method is due to phase fluctuations in the wave (forward scattering). However, we should then see how phase fluctuations can be excluded, since they have no physical meaning and do not correspond to the energy losses in a random medium. The answer is very simple in a one-dimensional problem: we must employ a coordinate system that moves with the fluctuation velocity without phase fluctuations. Let us consider again the nonlinear wave equation (3.1), but, for the sake of simplicity, ε will be assumed to be a function of t alone (space fluctuations were considered in [15]). We shall use a reference system that travels at an undetermined (as yet) velocity

$$x' = x - \int c(t) dt, \quad t' = t. \quad (4.1)$$

In the new variables, Eq. (4.1) takes the form (primes are omitted)

$$\frac{\partial^2 u}{\partial t^2} - 2c(t) \frac{\partial^2 u}{\partial t \partial x} - \frac{\partial c}{\partial t} \frac{\partial u}{\partial x} + [c^2 - (1 + \mu\varepsilon)^2] \frac{\partial^2 u}{\partial x^2} = 0. \quad (4.2)$$

Here we consider a single-wave approximation again and, in accordance with the many-scale method used to derive evolution equations [19, 20] (in fact, we used it in the previous section), we introduce along with the "fast" time the slow time $T = \mu^2 t$ and seek a solution in the form of asymptotic series that are accurate to within μ^2

$$u(x, t, T) = u^{(0)}(x, T) + \mu u^{(1)}(x, t, T) + \mu^2 u^{(2)}(x, t, T), \quad (4.3)$$

$$c(t) = 1 + \mu c^{(1)}(t) + \mu^2 c^{(2)}(t). \quad (4.4)$$

The principal terms in these series correspond to a wave heading to the right. The zeroth order of perturbation theory with respect to μ is satisfied automatically, while for the first order an equation for the scattered field is derived

$$\frac{\partial^2 u^{(1)}}{\partial t^2} - 2 \frac{\partial^2 u^{(1)}}{\partial t \partial x} = \frac{\partial c^{(1)}}{\partial t} \frac{\partial u^{(0)}}{\partial x} + 2 \left(\varepsilon - c^{(1)} \right) \frac{\partial^2 u^{(0)}}{\partial x^2}. \quad (4.5)$$

It can be integrated trivially taking into account the initial conditions of the absence of reflected waves as $t \rightarrow -\infty$

$$u^{(1)} = \int_0^\infty \frac{\partial u^{(0)}}{\partial x}(x+2\tau, T) \varepsilon(t-\tau) d\tau - \frac{\partial u^{(0)}}{\partial x}(x, T) \int_{-\infty}^t (\varepsilon(\tau) - c^{(1)}(\tau)) d\tau. \quad (4.6)$$

The last term in (4.6) is easily seen to correspond to forward scattering, and it results in an unrestricted increase in $u^{(1)2}$. To exclude this, we assume

$$c^{(1)}(t) = \varepsilon(t), \quad (4.7)$$

then the scattered field remains restricted. For the second order in μ we have an inhomogeneous linear equation for $u^{(2)}$

$$\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} - 2 \frac{\partial}{\partial x} \right) u^{(2)} = F(x, t, T), \quad (4.8)$$

$$F = 2 \frac{\partial^2 u^{(0)}}{\partial T \partial x} + \frac{\partial^2 (u^{(0)})^2}{\partial x^2} + 2\varepsilon \frac{\partial^2 u^{(1)}}{\partial t \partial x} + \frac{\partial \varepsilon}{\partial t} \frac{\partial u^{(1)}}{\partial x} + 2c^{(2)} \frac{\partial^2 u^{(0)}}{\partial x^2} + \frac{\partial c^{(2)}}{\partial t} \frac{\partial u^{(0)}}{\partial t}. \quad (4.9)$$

In order to keep $u^{(2)}$ restricted as $t \rightarrow \infty$, the following condition should be satisfied

$$\langle F \rangle = \lim_{2\Delta} \frac{1}{2\Delta} \int_{-\Delta}^{+\Delta} F(x, t, T) dt = 0. \quad (4.10)$$

It is this condition that results in the desired evolution equation for $u^{(0)}$ (index (0) is omitted)

$$\frac{\partial u}{\partial T} + \frac{\sigma^2}{2} \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} + \int_0^\infty W(\tau) \frac{\partial^2 u}{\partial x^2}(x+2\tau, T) d\tau = 0, \quad (4.11)$$

where W is the correlation function of ε , as previously. Equation (4.11) contains only determinate coefficients, as in the case of the average field method. However, since field averaging is performed in a reference system that moves at a random velocity and "tracking" fluctuations of the wave phase, then $u(x, T)$ will describe the average wave form rather than the average field. The average field is easily calculated by the formula

$$\langle u \rangle = \int_{-\infty}^\infty u^{(0)}(x-t-\Theta) \Gamma(\Theta, t) d\Theta, \quad (4.12)$$

where Γ is the probability distribution of phase Θ .

Therefore, we have developed a consistent scheme to describe the properties of nonlinear waves in random media. It is clear from this paper that the degree of completeness and generality obtained is less than that known for linear media. A number of generalizations of the scheme have been carried out [17, 18] but only for one-dimensional medium fluctuations. The issue of nonlinear wave-field behavior in two-dimensional and three-dimensional random media requires further investigation.

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