

EXPLODING SOLUTIONS FOR THREE-DIMENSIONAL RIMMING FLOWS

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Summary

We examine the linear stability of a thin film of viscous fluid on the inside of a cylinder with horizontal axis, rotating about this axis. Unlike previous models, both axial and azimuthal components of the hydrostatic pressure gradient are taken into account, which yields solutions which collapse in both dimensions. Two types of such solutions are found: disturbances with zero and non-zero net mass (the former have greater explosion rates, that is, their amplitudes grow faster than those of the latter). It is also shown that, despite the existence of exploding disturbances, all solutions with harmonic dependence on time (eigenmodes) are neutrally stable.

1. Introduction

While examining two-dimensional stability of a viscous film inside a rotating horizontal cylinder, Benilov, O'Brien and Sazonov (1) derived a linear equation with very unusual properties. It admits infinitely many stable solutions with harmonic dependence on time (eigenmodes)—which, however, are not representative of the stability of the system as a whole. It turned out that the eigenmodes coexist with 'exploding' solutions, which develop a singularity in a finite time. These solutions describe the initial stage of formation of a two-dimensional drop of fluid on the 'ceiling' of the cylinder; it has been further argued that they could be responsible for the instability of rimming flows observed experimentally in (2, 3), in settings with important industrial applications (such as rotational moulding and coating of fluorescent tubes).

An even more exotic equation was derived in (4) for a three-dimensional model of rimming flows: not only does it admit infinitely many stable eigenmodes, but the corresponding eigenfunctions form a complete set. Thus, an arbitrary initial condition can be represented by a series involving eigenmodes—but, despite these being stable, the series may still diverge giving rise to an 'explosion'.

Note, however, that, even though the settings examined in (1, 4) were multi-dimensional, the exploding solutions 'collapse' in a single direction only (the azimuthal and axial directions, respectively). In order to understand why this happens, observe that generally disturbances in a rimming flow collapse due to the hydrostatic pressure gradient, which destabilizes the film at the 'ceiling' of the cylinder. Note also that a real (three-dimensional) drop of fluid can only form if its neck 'snaps' in both the axial and azimuthal directions. Thus, to describe its formation, the axial *and* azimuthal components of the pressure gradient should be included in the model—whereas (1) and (4) take into account *either* the azimuthal *or* axial component, respectively.

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Another omission of (1, 4) is associated with how the amplitude A of the exploding solution is related to its width W . In most cases considered in the literature (including (1, 4)), it is of the form

$$A \sim W^{-\alpha}, \quad (1)$$

where the ‘explosion rate’ α shows how fast A grows as $W \rightarrow 0$. Note also that, in both (1) and (4), the explosion rate was found to be

$$\alpha = 1. \quad (2)$$

This result can be interpreted using the mass conservation law—which, for a one-dimensional collapse, implies

$$A \times W \times s = \text{const}, \quad (3)$$

where the constant s depends on the shape of the exploding disturbance. Then, the explosion rate (2) results from the mass conservation law (3).

Observe, however, that (1), (2) are not the only way to satisfy (3), as the latter would also hold if

$$s = 0, \quad \text{const} = 0,$$

with α remaining undetermined. In other words, disturbances with zero net mass may have different explosion rates! However, no such solutions have been found in (1, 4).

The present paper aims to remedy the two omissions of (1, 4) outlined above. First, we shall derive an equation which takes into account both components of the pressure gradient. Secondly, for this equation we shall find disturbances with zero net mass and show that they have greater explosion rates than the ones found previously. We shall also find similar solutions for the models examined in (1, 4).

In what follows, section 2 formulates the problem mathematically and section 3 uses a small-flux approximation to examine harmonic disturbances. In sections 4, 5, we examine exploding disturbances and clarify the extent to which they are consistent with the approximations used to obtain them.

2. Formulation of the problem

2.1 *The governing equation*

Consider a thin film of incompressible liquid on the inside of a cylinder of radius R , with a horizontal axis, which is rotating about this axis with constant angular velocity Ω (see Fig. 1). Cylindrical coordinates (r, θ, z) are used, and so h , the thickness of the film, depends on the azimuthal angle θ , axial coordinate z , and time t . In what follows, we shall use the following non-dimensional variables:

$$\begin{aligned} \hat{\theta} &= \theta, & \hat{z} &= \frac{\varepsilon^{1/2} z}{R}, & \hat{t} &= \Omega t, \\ \hat{\eta} &= \frac{h}{\varepsilon R} - \frac{h^2}{2\varepsilon R^2}, \end{aligned} \quad (4)$$

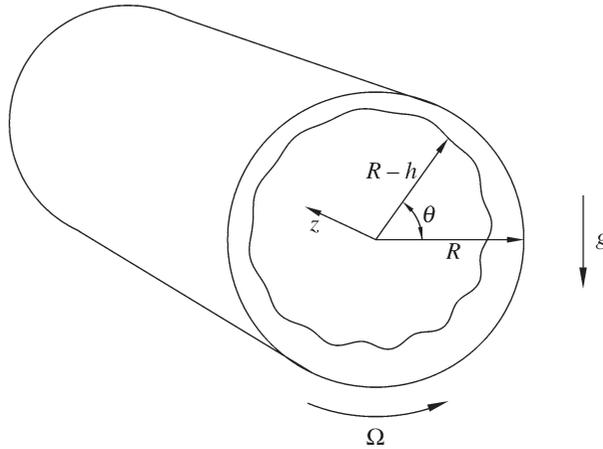


Fig. 1 Formulation of the problem: liquid film inside a rotating horizontal cylinder

where

$$\varepsilon = \left(\frac{\nu \Omega}{g R} \right)^{1/2}.$$

Note that, for thin films ($h \ll R$), the second term of (4) is small—thus, $\hat{\eta}$ is essentially the non-dimensional thickness.

From a physical viewpoint, there are five effects governing the film: viscosity, gravity, the axial and azimuthal components of the hydrostatic pressure gradient, and inertia (described by the material derivatives in the Navier–Stokes equations). The only approach to modelling these effects developed so far is based on the lubrication approximation, where the first effect is assumed dominant, the following two may or may not be dominant, and the last two must be treated as perturbations. Thus, depending on the parameters involved, various asymptotic equations can be derived. We are particularly interested in two such equations derived originally in (5). Omitting the hats of the non-dimensional variables, we can write these equations in the form (see (1, 4))

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \theta} \left[\eta - \frac{\eta^3}{3} \cos \theta \right] + \frac{\partial}{\partial \theta} \left(\frac{\varepsilon \eta^3}{3} \frac{\partial \eta}{\partial \theta} \sin \theta \right) = 0, \quad (5)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \theta} \left[\eta - \frac{\eta^3}{3} \cos \theta \right] + \frac{\partial}{\partial z} \left(\frac{\eta^3}{3} \frac{\partial \eta}{\partial z} \sin \theta \right) = 0. \quad (6)$$

The pair of terms in square brackets describe viscous entrainment of the film by the rotation of the cylinder and the effect of gravity, whereas the last terms of (5) and (6) describe the azimuthal and axial components of the hydrostatic pressure gradient, respectively. Note that, within the lubrication approximation, the θ -component of the pressure gradient is weak, that is, the last term in (5) is small (see (1)). The z -component of the pressure gradient, in turn, can be of leading order, that is, the last term in (6) can be as large as the other terms in this equation (see (4)).

In this paper, we shall examine the combined effect of the axial and azimuthal components of the hydrostatic pressure gradient. To derive the most general model, one should treat the former as a leading-order effect, whereas the latter should be treated as a perturbation. As a result, the following equation can be derived:

$$\begin{aligned} & \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \theta} \left\{ \eta - \frac{\eta^3}{3} \cos \theta + \varepsilon \left[\frac{\eta^3}{3} \frac{\partial \eta}{\partial \theta} \sin \theta - \frac{\eta^3}{2} \left(\frac{\partial \eta}{\partial z} \right)^2 \cos \theta - \frac{5\eta^4}{24} \frac{\partial^2 \eta}{\partial z^2} \cos \theta \right] \right\} \\ & + \frac{\partial}{\partial z} \left\{ \frac{\eta^3}{3} \frac{\partial \eta}{\partial z} \sin \theta + \varepsilon \left[\frac{3\eta^5}{5} \frac{\partial^3 \eta}{\partial z^3} \sin \theta + 4\eta^4 \frac{\partial \eta}{\partial z} \frac{\partial^2 \eta}{\partial z^2} \sin \theta + \frac{7\eta^3}{3} \left(\frac{\partial \eta}{\partial z} \right)^3 \sin \theta \right. \right. \\ & \left. \left. - \frac{19\eta^4}{24} \frac{\partial^2 \eta}{\partial z \partial \theta} \cos \theta - \frac{11\eta^3}{6} \frac{\partial \eta}{\partial \theta} \frac{\partial \eta}{\partial z} \cos \theta + \frac{35\eta^4}{24} \frac{\partial \eta}{\partial z} \sin \theta \right] \right\} = 0 \end{aligned} \quad (7)$$

(an Appendix with a derivation of this equation can be obtained from the authors of this paper on request). Observe that equation (7) cannot be obtained through a naive ‘amalgamation’ of (5) and (6), as it describes not only the two components of the hydrostatic pressure, but also their interaction.

2.2 Steady states and disturbances

Let the solution of equation (7) be independent of t and z , that is, $\eta(\theta, z, t) = \bar{\eta}(\theta)$. Then (7) yields

$$\bar{\eta} - \frac{1}{3} \bar{\eta}^3 \cos \theta + \frac{1}{3} \varepsilon \bar{\eta}^3 \frac{\partial \bar{\eta}}{\partial \theta} \sin \theta = q, \quad (8)$$

where q is a constant of integration (physically, q is the non-dimensional mass flux). If $\varepsilon = 0$, this equation has a smooth solution for $q < \frac{2}{3}$ —see (6, 7), while the limit $\varepsilon q^3 \ll 1$ has been examined in (3, 8, 9). Note that the latter assumption is a part of the lubrication approximation, which has already been used in the derivation of (8).

In order to examine $\bar{\eta}$ for stability, assume that

$$\eta(\theta, z, t) = \bar{\eta}(\theta) + \eta'(\theta, z, t), \quad (9)$$

where η' represents the disturbance. Substituting (9) into (8) and omitting the nonlinear terms, we obtain (with primes omitted)

$$\begin{aligned} & \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \theta} \left\{ \eta - \bar{\eta}^2 \eta \cos \theta + \varepsilon \left[\left(\bar{\eta}^2 \frac{\partial \bar{\eta}}{\partial \theta} \eta + \frac{\bar{\eta}^3}{3} \frac{\partial \eta}{\partial \theta} \right) \sin \theta - \frac{5\bar{\eta}^4}{24} \frac{\partial^2 \eta}{\partial z^2} \cos \theta \right] \right\} \\ & + \frac{\partial}{\partial z} \left\{ \frac{\bar{\eta}^3}{3} \frac{\partial \eta}{\partial z} \sin \theta + \varepsilon \left[\left(\frac{3\bar{\eta}^5}{5} \frac{\partial^3 \eta}{\partial z^3} + \frac{35\bar{\eta}^4}{24} \frac{\partial \eta}{\partial z} \right) \sin \theta \right. \right. \\ & \left. \left. - \left(\frac{19\bar{\eta}^4}{24} \frac{\partial^2 \eta}{\partial z \partial \theta} + \frac{11\bar{\eta}^3}{6} \frac{\partial \bar{\eta}}{\partial \theta} \frac{\partial \eta}{\partial z} \right) \cos \theta \right] \right\} = 0. \end{aligned} \quad (10)$$

The main difficulty associated with this equation is that the explicit form of the coefficients is unknown; recall that $\bar{\eta}(\theta)$ is determined by (8). The only exception is the limit of small flux, $q \ll 1$, in which case (8) admits an explicit asymptotic solution:

$$\bar{\eta} = q + \frac{1}{3}q^3 \cos \theta + O(q^5) \quad \text{if } q \ll 1. \quad (11)$$

Physically, this limit corresponds to viscous forces dominating gravity—as a result, the asymmetry introduced by the latter is negligible, and the thickness of the film is almost constant. Note, however, that the small-flux assumption is not crucial to our study, as all of the results can be generalized for $q = O(1)$.

Now, substitute (11) into (10) and assume, for simplicity, that

$$\frac{\varepsilon q^3}{\Delta \theta} \gg q^4, \quad (12)$$

where $\Delta \theta$ is the characteristic azimuthal scale of the solution. Then, omitting the $O(q^4)$ terms and smaller, we obtain

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \theta} \left[\left(1 - q^2 \cos \theta \right) \eta + \frac{\varepsilon q^3 \sin \theta}{3} \frac{\partial \eta}{\partial \theta} \right] + \frac{\partial}{\partial z} \left(\frac{q^3 \sin \theta}{3} \frac{\partial \eta}{\partial z} \right) = 0. \quad (13)$$

Equation (13) describes azimuthal (θ -) propagation of disturbances and their diffusion in both θ - and z -directions. Accordingly, the factor $(1 - q^2 \cos \theta)$ is the propagation speed, whereas $-\frac{1}{3}\varepsilon q^3 \sin \theta$ and $-\frac{1}{3}q^3 \sin \theta$ are the effective diffusivities in the θ - and z -directions, respectively. Note that in the lower half of the cylinder ($-\pi < \theta < 0$), the diffusivities are positive, whereas in the upper half ($0 < \theta < \pi$), they are both negative (because of ‘inverse’ gravity).

Finally, note that the lubrication approximation holds if the θ -diffusion term in (13) is much smaller than the propagation term,

$$\left| \frac{\partial}{\partial \theta} \left(\frac{\varepsilon q^3 \sin \theta}{3} \frac{\partial \eta}{\partial \theta} \right) \right| \ll \left| \frac{\partial \eta}{\partial \theta} \right|,$$

or, equivalently,

$$\frac{\varepsilon q^3}{\Delta \theta} \ll 1. \quad (14)$$

One can see that since q is small, conditions (12) and (14) are consistent.

3. Harmonic disturbances

In this section we shall examine the eigenmodes, that is, solutions with harmonic dependence on time. We shall further assume that the dependence on the axial variable is also harmonic, that is,

$$\eta(\theta, z, t) = \phi(\theta) e^{ikz - i\omega t}, \quad (15)$$

where ω is the frequency and k is the axial wavenumber. Substitution of (15) into (13) yields

$$\frac{d}{d\theta} \left[\left(1 - q^2 \cos \theta \right) \phi + \varepsilon (\sin \theta) \frac{d\phi}{d\theta} \right] - (i\omega + \kappa^2 \sin \theta) \phi = 0, \quad (16)$$

where

$$\epsilon = \frac{\epsilon q^3}{3}, \quad \kappa^2 = \frac{k^2 q^3}{3}. \tag{17}$$

Equation (16) and the periodicity condition

$$\phi(\theta + 2\pi) = \phi(\theta) \tag{18}$$

form an eigenvalue problem, where $\phi(\theta)$ is the eigenfunction and ω is the eigenvalue. If $\text{Im } \omega > 0$ the film is unstable.

For $\kappa^2 = 0$, problem (16) to (18) has been examined in (1), and the same approach can be used in the present case. Thus, we shall solve (16) to (18) asymptotically, using a WKB-type method based on the smallness of ϵ (we could also take into account that $q^2 \ll 1$, but this would not make the problem much simpler).

The most general asymptotic solution can be obtained when $\kappa^2 \sim \epsilon^{-1}$, which actually includes the limits $\kappa^2 \ll \epsilon^{-1}$ and $\kappa^2 \gg \epsilon^{-1}$ as well. Then, introducing

$$\kappa^2 = \epsilon \kappa'^2, \quad \Omega = \epsilon \omega, \tag{19}$$

we seek a solution in the form

$$\phi(\theta) = \exp \left[\frac{1}{\epsilon} \int_0^\theta \psi(\theta') \, d\theta' \right], \tag{20}$$

where $\psi(\theta)$ is the new unknown function. Substitution of (19), (20) into (16) yields

$$(\sin \theta) \psi^2 + (1 - q^2 \cos \theta) \psi - i\Omega - \kappa'^2 \sin \theta + \epsilon \left[(\cos \theta) \psi + (\sin \theta) \frac{d\psi}{d\theta} + q^2 \sin \theta \right] = 0. \tag{21}$$

Next, seek a solution in the form

$$\psi = \psi^{(0)} + \epsilon \psi^{(1)} + \dots \tag{22}$$

To leading order, (21) yields

$$\psi^{(0)2} \sin \theta + (1 - q^2 \cos \theta) \psi^{(0)} - i\Omega - \kappa'^2 \sin \theta = 0.$$

This quadratic equation has two solutions for $\psi^{(0)}$, one of which is singular at $\theta = 0, \pi$ and must be discarded. The other one is

$$\psi^{(0)} = \frac{q^2 \cos \theta - 1 + \sqrt{(1 - q^2 \cos \theta)^2 + 4(i\Omega + \kappa'^2 \sin \theta) \sin \theta}}{2 \sin \theta}, \tag{23}$$

where the branch of square root is such that $\sqrt{1} = 1$. Substituting the leading-order eigenfunction (20), (23) into the periodicity condition (18) and using (19) to return to ω and κ^2 , we obtain the following equation for the leading-order eigenvalue:

$$\int_0^{2\pi} \frac{q^2 \cos \theta - 1 + \sqrt{(1 - q^2 \cos \theta)^2 + 4\epsilon(i\omega + \kappa^2 \sin \theta) \sin \theta}}{2\epsilon \sin \theta} \, d\theta = 2\pi i n, \tag{24}$$

where n is an integer (the mode number). Equation (24) can be rearranged as

$$F(\omega) = n, \tag{25}$$

where

$$F(\omega) = \int_0^{\pi/2} \frac{\sqrt{F_{1-} + iF_2} - \sqrt{F_{1+} - iF_2} + \sqrt{F_{1+} + iF_2} - \sqrt{F_{1-} - iF_2}}{4\pi i\epsilon \sin \theta} d\theta,$$

$$F_{1\pm} = \left(1 + q^2 \cos \theta\right)^2 \pm 4\epsilon\kappa^2 (\sin \theta)^2, \quad F_2 = 4\epsilon\omega \sin \theta.$$

It can be shown that

- if $\text{Im } \omega \neq 0$, then $\text{Im } F(\omega) \neq 0$ —hence, (25) does not have any complex roots;
- for real ω , $F(\omega)$ is a real, monotonically growing function, such that

$$\begin{aligned} F(\omega) &= 0 && \text{if } \omega = 0, \\ F(\omega) &\rightarrow \infty && \text{as } \omega \rightarrow \infty. \end{aligned}$$

Hence, for any n , (25) has a unique *real* root, $\omega = \omega_n$ —which means that, to leading order, all disturbances are neutrally stable. Furthermore, it can be similarly demonstrated that, in the next order, the eigenvalue ω also remains real (neutral stability).

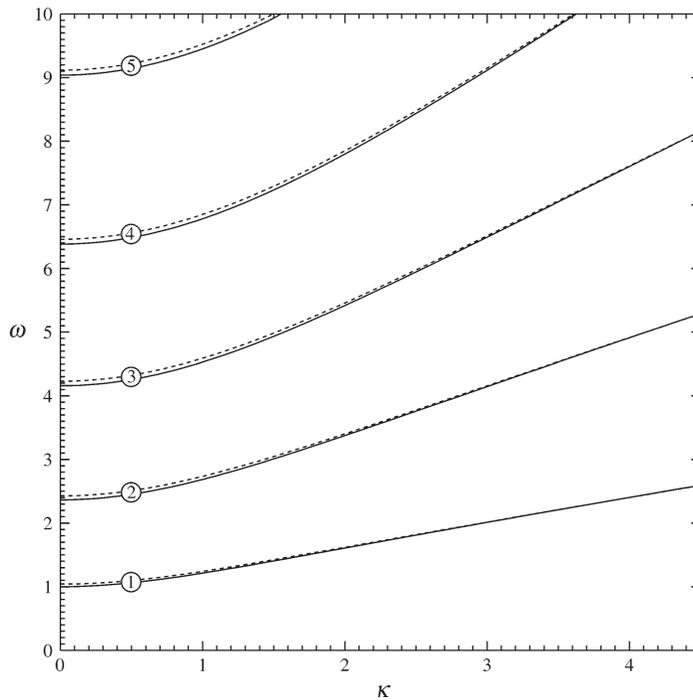


Fig. 2 The solution of the eigenvalue problem (16) to (18) for $\epsilon = q^2 = 0.3$. The numerical solution is represented by the solid line and the asymptotic solution computed (using (25)) is represented by the dotted line. The curves are marked by the corresponding mode number

The results obtained through the asymptotic equation (25) have been compared to the direct numerical solution of problem (16) to (18). For all physically relevant values of parameters ($\epsilon, q^2 \leq 0.1$) the difference between the asymptotic and exact solutions is hardly visible. Thus, Fig. 2 shows the results for $\epsilon = q^2 = 0.3$, in which case the agreement between the two solutions is still very good.

4. Exploding solutions

It turns out that despite all the eigenmodes being stable, equation (13) admits an exploding solution, which develops a singularity in a finite time.

Following (1), we change to a co-moving reference frame,

$$y = \theta - t, \quad z = z, \quad t = t, \quad (26)$$

and re-scale the coordinates,

$$y = \sqrt{\epsilon} \tilde{y}, \quad z = \sqrt{\frac{q^3}{3}} \tilde{z}, \quad t = t. \quad (27)$$

The new variables imply that we deal with a narrow pulse advancing along the inner surface of the cylinder in a counter-clockwise direction, at a unit angular speed. Then, substituting (26), (27) into (13), we omit tildes and terms involving the small parameters ϵ and q ,

$$\frac{\partial \eta}{\partial t} + (\sin t) \left(\frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2} \right) = 0. \quad (28)$$

Rewriting this equation in polar variables and seeking a radially symmetric solution, one can reduce (28) to

$$\frac{\partial \eta}{\partial t} + \frac{\sin t}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \eta}{\partial \rho} \right) = 0, \quad (29)$$

where $\rho = \sqrt{y^2 + z^2}$.

Now, seek a solution of the form

$$\eta(\rho, t) = A(t)f(\zeta), \quad \zeta = \frac{\rho}{W(t)}, \quad (30)$$

where $A(t)$ and $W(t)$ are the amplitude and width of the pulse, and $f(\zeta)$ describes its shape. To ensure that η is smooth at the origin, we must require

$$\frac{df}{d\zeta} = 0 \quad \text{at } \zeta = 0. \quad (31)$$

Without loss of generality, we can assume that

$$f = 1 \quad \text{at } \zeta = 0. \quad (32)$$

Substitution of (30) into (29) yields

$$\frac{d^2 f}{d\xi^2} + \left(\frac{1}{\xi} + \zeta \right) \frac{df}{d\xi} + \alpha f = 0, \quad (33)$$

$$\frac{1}{A} \frac{dA}{dt} = -\frac{\alpha}{W} \frac{dW}{dt}, \quad \frac{dW}{dt} = -\frac{\sin t}{W}, \quad (34)$$

where the constant α has been introduced in the course of separation of variables (the second such constant can be eliminated by re-scaling W and ξ). Equations (34) can be readily solved,

$$W = \sqrt{W_0^2 - 4(\sin \frac{1}{2}t)^2}, \quad (35)$$

$$A = \left(\frac{W}{W_0} \right)^{-\alpha}, \quad (36)$$

where it is implied that the pulse's initial amplitude and width are

$$W(0) = W_0, \quad A(0) = 1,$$

respectively.

Equations (35), (36) show that the evolution of the pulse depends on whether or not the initial width W_0 exceeds the threshold value of 2.

- If $W_0 > 2$, the solution is smooth at all times. Note that, between $t = 0$ and $t = \pi$, the pulse is travelling through the upper half of the cylinder, where the diffusivity is *negative*. Accordingly, the width of the pulse is decreasing and the amplitude is growing. At $t = \pi$, the pulse enters the region of *positive* diffusivity, and by the time it reaches the starting point ($t = 2\pi$), it restores its initial parameters. This cycle repeats itself indefinitely.
- If $W_0 \leq 2$, one can see that

$$W(t) \rightarrow 0, \quad A(t) \rightarrow \infty \quad \text{as } t \rightarrow t_e,$$

where

$$t_e = 2 \arcsin(\frac{1}{2}W_0)$$

is the time of 'explosion'. Thus, if the pulse is sufficiently narrow initially, it blows up due to the effect of 'anti-diffusivity' before it leaves the upper half of the cylinder. Observe that α determines how quickly the amplitude of the pulse tends to infinity as $W \rightarrow 0$ (see (36))—accordingly, we shall refer to this parameter as the explosion rate.

Before we examine the shape of the pulse, $f(\xi)$, observe that equation (29) conserves the net mass,

$$\int_0^\infty \eta(\rho, t) \rho d\rho = \text{const}, \quad (37)$$

which imposes certain restrictions on $f(\xi)$ and the explosion rate α . Indeed, substituting (30) into (37), we obtain

$$A(t)W^2(t) \int_0^\infty f(\xi) \xi d\xi = \text{const}. \quad (38)$$

Then, comparing (38) and (36), we see that either

$$\alpha = 2, \tag{39}$$

or

$$\int_0^\infty f(\xi)\xi d\xi = 0. \tag{40}$$

Note that these restrictions could also be obtained directly from (33) under the assumption that the mass integral converges.

If $\alpha = 2$, it can be verified by inspection that the solution to (31) to (33) is

$$f(\xi) = \exp(-\frac{1}{2}\xi^2). \tag{41}$$

Note that this solution does not have to satisfy restriction (40)—and it does *not* satisfy it, that is, (41) corresponds to a non-zero net mass.

Generally, equation (33) can be reduced to Kummer’s or Whittaker’s equations (see (10)), whence one can demonstrate that problem (31) to (33), (41) has a solution only for $\alpha > 2$. Several examples of such $f(\xi)$ can be seen in Fig. 3—naturally, they are sign-variable. Thus, the main pulse of such a disturbance is surrounded by a ‘ring’ of the opposite sign. Interestingly, rings of greater amplitude (depth) give rise to higher explosion rates.

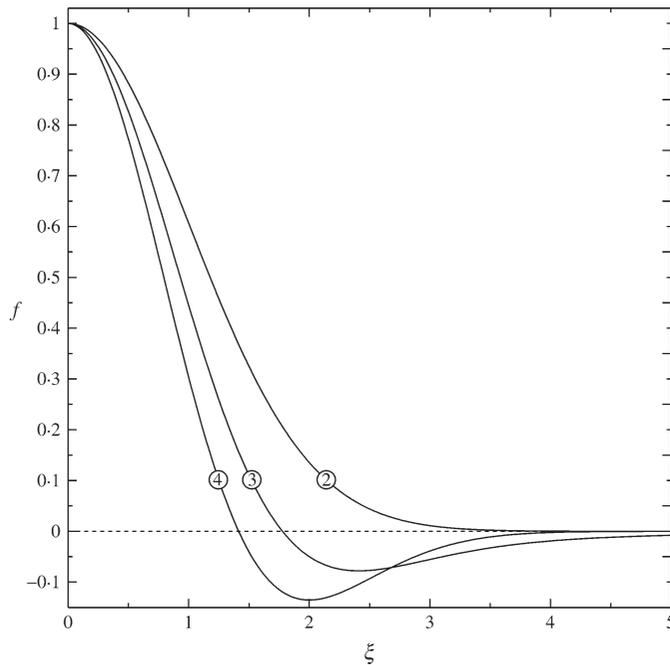


Fig. 3 The shapes of exploding disturbances (the solution of the boundary-value problem (31) to (33)). The curves are marked with the corresponding values of the explosion rate α . The case $\alpha = 2$ corresponds to disturbance with non-zero net mass

Finally, note that the exploding solutions found in (1, 4) have non-zero net mass (that is, they are analogous to the present solution with $\alpha = 2$). Solutions with zero net mass, and greater explosion rates, are considered in Appendix.

5. Discussion

5.1 *The applicability of exploding solutions*

When dealing with short-wave asymptotic solutions (such as our exploding solutions), it is always a concern whether they comply with the assumptions under which they have been derived. In the present context, we need to clarify to what extent solution (30) is consistent with the approximations used to derive it. There were three of those:

1. the lubrication approximation, using which the Navier–Stokes equations have been reduced to equation (7),
2. the small-flux approximation, from which equation (13) was obtained,
3. the short-wave approximation from which we reduced (13) to (28).

Clearly, there is nothing inconsistent with the third link of our asymptotic ‘chain’, as the short-wave approximation actually improves when the pulse is approaching explosion. The situation with the other two approximations is less clear.

First of all, we shall backtrack substitutions (26), (27) and note that, in terms of the original non-scaled variables, the azimuthal width of the pulse is

$$\Delta\theta = \epsilon^{1/2}W(t).$$

Then, we substitute $\Delta\theta$ into criteria (12), (14) of the small-flux/lubrication approximations,

$$1 \gg \frac{\epsilon q^3}{\epsilon^{1/2}W(t)} \gg q^4.$$

Taking into account (17), we obtain

$$\epsilon^{1/2} \ll W(t) \ll \epsilon^{1/2}q^{-4}, \quad (42)$$

where both ϵ and q are small parameters.

The second inequality of (42) restricts non-exploding asymptotic solutions ($W_0 > 2$), as it does not allow the width of the pulse to be too large. Note, however, that this restriction is not essential, as all our results can be generalized for $q = O(1)$.

The first inequality of (42), in turn, restricts the slope of the film’s surface (in accordance with the lubrication approximation). As a result, an exploding solution ($W_0 \leq 2$) can be trusted only while its width $W(t)$ remains sufficiently large—which certainly excludes the time of explosion (and some period of time before that, of course).

In order to understand the physical meaning of the exploding solutions, imagine a short-scale perturbation on the surface of the film. When the rotation of the cylinder turns it ‘upside down’, gravity starts increasing its amplitude and/or shortening its size. Once the perturbation is sufficiently large and narrow, a drop of fluid should detach itself from the film and fall down.

It can be conjectured that our asymptotic solution describes an initial stage of the formation of a drop. Note, however, that, once the first inequality in condition (42) is violated, nonlinear effects take over and the asymptotic solution becomes invalid.

5.2 *The effect of surface tension*

The joint effect of surface tension and the azimuthal component of the pressure gradient has been examined in (11). It has been shown that surface tension

- destabilizes some of the short-wave harmonic disturbances, and
- transforms the exploding solutions into transient ones (which initially grow, but eventually decay).

Similar conclusions have been drawn for the interaction of surface tension with the axial component of the pressure gradient (12, 13)—although the instability was found for *long*-wave disturbances and shown to be very weak. Thus, in the full problem involving all three effects, one should expect strong short-wave (harmonic) instability. We shall not discuss this problem in further detail, but note only that it may not be examined using the lubrication approximation (which is violated by the short-wave disturbances—see (11)).

6. Concluding remarks

In this paper, we address two important issues that have not been considered in previous work. First, we present a model of rimming flows (equation (7)), which includes both axial and azimuthal components of the pressure gradient. Accordingly, linear stability analysis yields solutions which collapse in both directions and describe (an early stage of) a drop formation. Secondly, a new type of exploding solutions has been found—namely, disturbances with zero net mass. These solutions consist of a main pulse, surrounded by a ring of the opposite sign. Interestingly, the amplitude of the ring is related to the explosion rate of the solution as a whole: a larger ring gives rise to a stronger explosion. Note also that similar solutions exist, but have been missed, in the models examined in (1, 4)—this omission is corrected in the present paper (see the Appendix). Most importantly, in all three cases, the ‘new’ solutions have greater explosion rates than those of disturbances with non-zero net mass. Nevertheless, despite the existence of exploding disturbances, all solutions with harmonic dependence on time (eigenmodes) are neutrally stable. This paradoxical conclusion is a characteristic property of rimming flows, as it has been obtained for all models not including surface tension; see (3, 1, 4).

Note, finally, that even though our results have been obtained using the small-flux approximation (11), they can be generalised for an arbitrary value of the flux q (although the algebra involved would be extremely cumbersome).

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APPENDIX

Exploding solutions with zero net mass for the models considered in (1, 4)

In (1), exploding solutions were described by

$$\frac{\partial \eta}{\partial t} + (\sin t) \frac{\partial^2 \eta}{\partial y^2} = 0 \quad (\text{A1})$$

(compare (A1) with (29)). Then, assuming that

$$\eta(y, t) = A(t) f(\xi), \quad \xi = \frac{y}{W(t)}, \quad (\text{A2})$$

one can verify that the shape of the pulse satisfies

$$\frac{d^2 f}{d\xi^2} + \xi \frac{df}{d\xi} + \beta f = 0, \quad (\text{A3})$$

where β is a separation-of-variables constant (it is an equivalent of α introduced in the main body of the paper). The expressions for width W and amplitude A of the pulse are the same as before (that is, given by (35), (36) with α replaced by β).

Note also that equation (A1) conserves net mass, that is,

$$\int_{-\infty}^{\infty} \eta(y, t) dy = \text{const.} \quad (\text{A4})$$

Substitution of (A2) into (A4) shows that either

$$\beta = 1 \quad (\text{A5})$$

or

$$\int_0^{\infty} f(\xi) d\xi = 0. \quad (\text{A6})$$

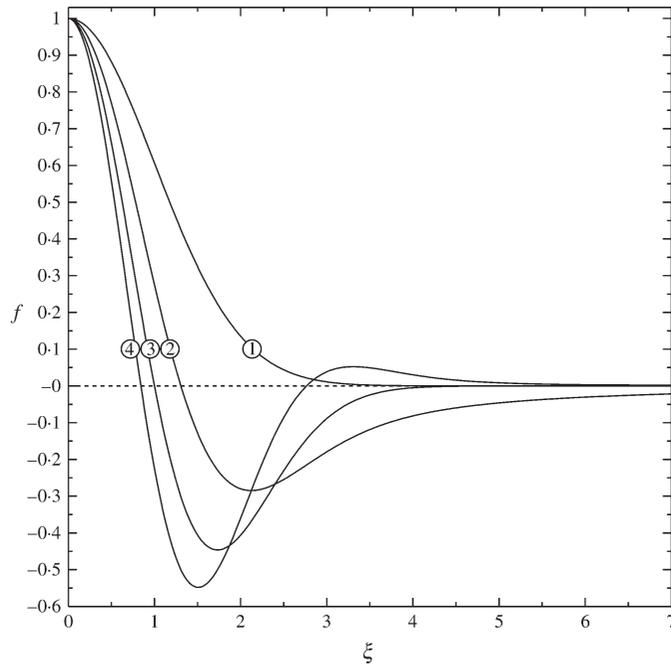


Fig. 4 The shapes of exploding disturbances for the problems examined originally in (1, 4) (the solution of the boundary-value problem (A3), (31), (32)). The curves are marked with the corresponding values of the explosion rate β . The case $\beta = 1$ corresponds to disturbance with non-zero net mass

For $\beta = 1$, it can be verified by inspection that the solution is exactly the same as the one found in the main body of the paper for $\alpha = 2$, that is, given by (41). For other values of β , (A3) can be reduced to Kummer’s or Whittaker’s equations (just like its predecessor, (33)), and it can be shown that problem (A3), (31), (32), (A6) has solutions only for $\beta > 1$. Several examples of these are shown in Fig. 4—one can see that, again, a ring of the opposite sign surrounds the main pulse and gives rise to a higher explosion rate.

Finally, we shall briefly discuss the exploding solutions examined in (4), which were described by

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial \theta} + (\sin \theta) \frac{\partial^2 \eta}{\partial z^2} = 0.$$

For this equation, the equivalent of substitution (A2) is

$$\eta(\theta, z, t) = A(\theta, t) f(\xi), \quad \xi = \frac{z}{W(\theta, t)}.$$

Then, A and W satisfy

$$\frac{1}{A} \left(\frac{\partial A}{\partial t} + \frac{\partial A}{\partial \theta} \right) = -\frac{\beta}{W} \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial \theta} \right),$$

$$W \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial \theta} \right) + \sin \theta = 0,$$

whereas $f(\xi)$ satisfies the same equation as in the previous case (that is, equation (A3)). As a result, even though the width and amplitude of the pulse are different from the previous case, the pulse shape and explosion rate remain exactly the same.