Thick drops on a slowly oscillating substrate

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(Received 16 March 2013; revised manuscript received 23 May 2013; published 14 August 2013)

We examine the evolution of a liquid drop on an inclined substrate oscillating vertically. The oscillations are weak and slow, which makes the liquid's inertia and viscosity negligible (so that the drop's shape is determined by a balance of surface tension, gravity, and vibration-induced inertial force). No assumptions are made about the drop's thickness, which extends our previous results on thin drops [Benilov, Phys. Rev. E **84**, 066301 (2011)] to more realistic situations. It is shown that, if the amplitude of the substrate's oscillations exceeds a certain threshold value ε_*, the drop climbs uphill. ε_*, however, strongly depends on the thickness of the drop, which, in turn, depends on the liquid's equilibrium contact angle $\bar{\beta}$. In particular, there is a dramatic decrease in ε_* when $\bar{\beta}$ exceeds a certain threshold, which means that thick drops climb uphill for a much weaker vibration of the substrate. At the same time, the frequency range of the substrate's vibration within which drops climb uphill becomes much narrower.

DOI: 10.1103/PhysRevE.88.023013 PACS number(s): 47.55.D−, 47.55.nb

I. INTRODUCTION

Recent experiments reported in Refs. [1,2] demonstrated that drops on a vibrating inclined substrate can climb uphill. The results of Ref. [3], in turn, showed that drops on a horizontal substrate vibrating at an angle to itself can be driven in a given direction and at a given speed by "tuning" the phase difference between the vertical and horizontal components of the vibration.

Three attempts have been made to model the observed effect. Two-dimensional thin drops were examined in Refs. [4,5] in the limits of low and high Reynolds numbers, respectively. *Three*-dimensional thin drops were examined in Ref. [6] under the assumptions that the vibration is weak and the drop is quasistatic (i.e., its shape is determined by the balance of surface tension, gravity, and the inertial force due to the vibration). All three models suggested that, if the amplitude ε and frequency ω of the vibration are within a certain region on the (ε,ω) plane, the drop climbs uphill.

Still, despite qualitative agreement, the experimental and theoretical results cannot be compared quantitatively, as all of the latter were obtained for *thin* drops, whereas the former dealt with thick ones (i.e., their vertical and horizontal scales were comparable). Furthermore, it appears to be extremely difficult to make a thin drop climb uphill [7]; even though this may be possible in principle, it probably requires an impractically strong vibration.

The present paper extends the results of Ref. [6] to *thick* drops, so that the main obstacle to the comparison of theoretical and experimental results is now removed. We still use the quasistatic approximation and assume that the vibration is weak, but these assumptions are far less restrictive and can be satisfied in an experiment.

This paper is structured as follows: In Sec. II, the problem is formulated mathematically, then Secs. III and IV present an asymptotic calculation of the rise velocity of a drop on a vibrating substrate. Finally, the physical aspects of the results obtained are discussed in Sec. V.

II. FORMULATION

Consider a drop of liquid of density ρ, kinematic viscosity ν, and surface tension σ, on a vibrating substrate inclined at an angle α to the horizontal (see Fig. 1). We assume the substrate's vibration to be vertical, so the acceleration g due to gravity, whichever equation or boundary condition it normally appears in, can be replaced by an effective acceleration $a(t)$ due to both gravity and the vibration-induced inertial force.

Let t be the time variable and (r,θ,ϕ) be the standard set of spherical coordinates. The shape of the drop's free surface can then be described by the equation

$$r = R(\theta,\phi,t).$$

We shall use the quasistatic approximation assuming that the liquid's inertia and viscosity are weak. Then, the drop's shape is determined by the balance of surface tension, gravity, and vibration-induced inertial force, whereas its motion is driven by the contact lines. This approach has been previously used in Refs. [8,9] and, in the latter paper, it was derived from the Navier-Stokes equations under the assumption that the frequency of the substrate's vibration is sufficiently low.

Under the quasistatic approximation, R satisfies the equation

$$\sigma C + \rho a\, R\,(\sin\theta\cos\phi\sin\alpha + \cos\theta\cos\alpha) = P, \qquad (1)$$

where the curvature C of the surface is given by

$$C = \frac{\sin\theta}{R\left[R^2\sin^2\theta + \left(\frac{\partial R}{\partial\theta}\right)^2\sin^2\theta + \left(\frac{\partial R}{\partial\phi}\right)^2\right]^{3/2}}\Bigg[2R^3\sin^2\theta - R^2\left(\frac{\partial R}{\partial\theta}\sin\theta\cos\theta + \frac{\partial^2 R}{\partial\theta^2}\sin^2\theta + \frac{\partial^2 R}{\partial\phi^2}\right) + 3R\left(\frac{\partial R}{\partial\theta}\right)^2\sin^2\theta + 3R\left(\frac{\partial R}{\partial\phi}\right)^2 - \left(\frac{\partial R}{\partial\theta}\right)^3\sin\theta\cos\theta$$

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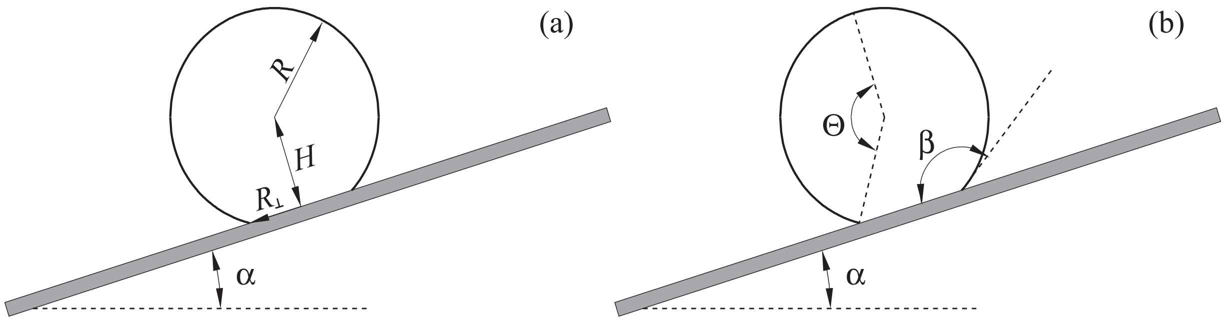

FIG. 1. The cross section of a drop on an inclined substrate. Spherical coordinates (r,θ,ϕ) are used. (a) $r = R(\theta,\phi,t)$ describes the drop's shape, H is the distance between the origin of the coordinate system and the substrate, and $R_\perp(\phi,t)$ is the local radius of the drop's base (i.e., the distance between the origin's projection onto the substrate and the point of the contact line, corresponding to the azimuthal angle ϕ). (b) $\beta(\phi,t)$ is the local contact angle, and $\theta = \Theta(\phi,t)$ describes the shape of the drop's base.

$$-2\frac{\partial R}{\partial\theta}\left(\frac{\partial R}{\partial\phi}\right)^2\cot\theta$$
$$-\frac{\partial^2 R}{\partial\phi^2}\left(\frac{\partial R}{\partial\theta}\right)^2+2\frac{\partial R}{\partial\theta}\frac{\partial R}{\partial\phi}\frac{\partial^2 R}{\partial\phi\partial\theta}-\frac{\partial^2 R}{\partial\theta^2}\left(\frac{\partial R}{\partial\phi}\right)^2\Bigg], \tag{2}$$

and the function $P(t)$ on the right-hand side of Eq. (1) is, physically, the pressure at the drop's center. Mathematically, P is an unknown and will need to be solved for. We note that Eq. (1) reflects the static balance of the capillary and hydrostatic contributions (the first and second terms, respectively) to the pressure field.

The contact line, i.e., the curve of intersection of the free surface and the substrate, is described by the equation

$$\theta = \Theta(\phi,t),$$

where Θ can be related to R and the distance H from the origin of the coordinate system to the substrate (see Fig. 1),

$$(R)_{\theta=\Theta}\cos\Theta = -H. \tag{3}$$

Note that H can be chosen at our discretion and may depend on t. We shall fix it later, with a view to make our perturbation expansion as simple as possible.

It is worth mentioning here that, even though the problem at hand is physically similar to its thin-drop counterpart [6], the latter is much simpler mathematically. For thin drops, the coefficients of the equations arising in the perturbation expansion depend only on r, whereas those in the present case depend on both r and θ. Given the complexity of the algebra involved in the present case, the simplifications introduced by choosing an appropriate $H(t)$ make a lot of difference.

The horizontal position of the origin can also be chosen at our discretion. In turns out that the most convenient choice is such that the origin's projection onto the substrate coincides with the centroid of the drop's base. To express this condition mathematically, we introduce the radial coordinate within the drop's base, $r_\perp = r\sin\theta$, and the local radius $R_\perp(\phi,t)$ of the base (see Fig. 1) related to H and Θ by

$$R_\perp = -H\tan\Theta. \tag{4}$$

Then, to fix the horizontal position of the origin, we require

$$\int_0^{2\pi}\int_0^{R_\perp} x r_\perp dr_\perp\, d\phi = 0, \qquad \int_0^{2\pi}\int_0^{R_\perp} y r_\perp dr_\perp\, d\phi = 0,$$

where x and y are the Cartesian coordinates in the plane of the substrate. Taking into account condition (4) and also that $x = r_\perp\cos\phi$, $y = r_\perp\sin\phi$, we obtain

$$\int_0^{2\pi}\tan^3\Theta\cos\phi\, d\phi = 0, \qquad \int_0^{2\pi}\tan^3\Theta\sin\phi\, d\phi = 0. \tag{5}$$

Since the coordinate system is moving with the drop, the origin's velocity is effectively the velocity of the drop. Its component normal to the substrate can be obtained by differentiating $H(t)$, whereas the other two components should be treated as independent unknowns. They will be denoted by $V_x(t)$ and $V_y(t)$, corresponding to the up-the-slope direction and lateral direction, respectively.

Following numerous previous researchers (e.g., Refs. [10,11]), we assume that the local normal velocity of the contact line depends solely on the local contact angle $\beta(\phi,t)$, which amounts to

$$\frac{\partial R_\perp}{\partial t}+V_x\left(\cos\phi+\frac{1}{R_\perp}\frac{\partial R_\perp}{\partial\phi}\sin\phi\right)+V_y\left(\sin\phi-\frac{1}{R_\perp}\frac{\partial R_\perp}{\partial\phi}\cos\phi\right)=v(\beta)\sqrt{1+\left(\frac{1}{R_\perp}\frac{\partial R_\perp}{\partial\phi}\right)^2}, \tag{6}$$

where $v(\beta)$ is a given function determined by the properties of the liquid and substrate. The local contact angle is given by

$$\sin\beta = \left\{\frac{\sin\Theta-\frac{1}{R}\frac{\partial R}{\partial\theta}\cos\Theta+\frac{1}{R\sin\Theta}\frac{\partial R}{\partial\phi}\frac{1}{R_\perp}\frac{\partial R_\perp}{\partial\phi}}{\sqrt{\left[1+\left(\frac{1}{R_\perp}\frac{\partial R_\perp}{\partial\phi}\right)^2\right]\left[1+\left(\frac{1}{R}\frac{\partial R}{\partial\theta}\right)^2+\left(\frac{1}{R\sin\Theta}\frac{\partial R}{\partial\phi}\right)^2\right]}}\right\}_{\theta=\Theta}. \tag{7}$$

We shall assume that the contact-line law $v(\beta)$ is strictly monotonic and, thus, vanishes at a single point, $v(\bar{\beta}) = 0$, where $\bar{\beta}$ is the *equilibrium contact angle*. This simple model (used previously in Refs. [5,6,8,9]) works well for liquid-substrate combinations with narrow hysteresis intervals (such as those examined in Refs. [12,13]) and can also help one to understand qualitative aspects of the general case.

In what follows, it is convenient to expand the contact-line law about $\bar{\beta}$,

$$v = v'(\beta - \bar{\beta}) + \frac{v''}{2}(\beta - \bar{\beta})^2 + \cdots, \tag{8}$$

where v' and v'' should be treated as known constants. Finally, we require that, at all times, the drop's volume be equal to that of a spherical cap of a radius R_0 and contact angle $\bar{\beta}$, which amounts to

$$\int_0^{2\pi}\int_0^{\Theta} R^3 \sin\theta \, d\theta \, d\phi + \frac{1}{2}H^3 \int_0^{2\pi} \tan^2\Theta \, d\phi = 2\pi R_0^3\left(1 - \cos\bar{\beta} - \frac{1}{2}\cos\bar{\beta}\sin^2\bar{\beta}\right). \tag{9}$$

R_0 can be interpreted as the curvature radius of an unperturbed drop in the absence of gravity and vibration. Given a suitable initial condition for $R_\perp$ and a specific form of $H(t)$, Eqs. (1)–(9) fully determine R, V_x, V_y, P, and the rest of the unknowns.

We shall use the following nondimensional variables:

$$\hat{t} = \frac{v't}{R_0}, \quad \hat{V}_{x,y} = \frac{V_{x,y}}{v'}, \quad \hat{R} = \frac{R}{R_0}, \tag{10}$$

$$\hat{C} = CR_0, \quad \hat{R}_\perp = \frac{R_\perp}{R_0}, \quad \hat{H} = \frac{H}{R_0}, \quad \hat{P} = \frac{R_0 P}{\sigma}, \quad \hat{v} = \frac{v}{v'}, \quad \hat{a} = \frac{a}{a_0}. \tag{11}$$

Rewriting Eqs. (2)–(7) in terms of the nondimensional variables and omitting the hats, one can observe that the resulting equations look exactly the same as before. Equations (1), (8), and (9), in turn, do change after nondimensionalization and become (hats omitted)

$$C + \varepsilon a R(\sin\theta\cos\phi\sin\alpha + \cos\theta\cos\alpha) = P, \tag{12}$$

$$v = (\beta - \bar{\beta}) + \frac{\eta}{2}(\beta - \bar{\beta})^2 + \cdots, \tag{13}$$

$$\int_0^{2\pi}\int_0^{\Theta} R^3 \sin\theta \, d\theta \, d\phi + \frac{1}{2}H^3 \int_0^{2\pi} \tan^2\Theta \, d\phi = 2\pi\left(1 - \cos\bar{\beta} - \frac{1}{2}\cos\bar{\beta}\sin^2\bar{\beta}\right), \tag{14}$$

where

$$\varepsilon = \frac{\rho a_0 R_0^2}{\sigma}, \quad \eta = \frac{v''}{v'}. \tag{15}$$

In the next section, Eqs. (2)–(7) and (12)–(14) will be examined for the limit

$$\varepsilon \ll 1,$$

which physically means that gravity and inertial force are much weaker than surface tension.

III. THE ANALYSIS

Observe that, if the drop's shape is initially symmetric with respect to the vertical plane passing through the origin, i.e.. if

$$R(\theta, -\phi, t) = R(\theta, \phi, t), \quad \Theta(-\phi, t) = \Theta(\phi, t), \quad R_\perp(-\phi, t) = R_\perp(\phi, t), \tag{16}$$

the substrate's vibration does not break this symmetry. As a result, the second of conditions (3) is satisfied identically. It is furthermore clear that a strictly vertical vibration cannot move a symmetric drop laterally, hence,

$$V_y = 0. \tag{17}$$

It is convenient to subdivide $a(t)$ into the oscillating and constant parts, representing the substrate's acceleration and gravity respectively, i.e.,

$$a = a^{(0)}(t) + a^{(1)}, \tag{18}$$

where $a^{(0)}(t)$ is a periodic function with a zero mean and $a^{(1)} > 0$ is a constant. Following the experiments [1,2], we assume

$$|a^{(0)}| \gg a^{(1)}. \tag{19}$$

Since a was nondimensionalized by its own magnitude, i.e., $a^{(0)} \sim 1$, condition (19) can be conveniently quantified by assuming

$$a^{(1)} = O(\varepsilon).$$

Given the periodicity of $a(t)$, it is clear that, after an initial adjustment, the drop's motion will also become periodic. Hence, we shall seek a *periodic* solution of the governing equations.

The solution of Eqs. (2)–(7) and (12)–(14), subject to conditions (16)–(18), will be sought in the form

$$R = R^{(0)} + \varepsilon R^{(1)} + \cdots, \quad P = P^{(0)} + \varepsilon P^{(1)} + \cdots,$$
$$\Theta = \Theta^{(0)} + \varepsilon\Theta^{(1)} + \cdots, \quad \beta = \beta^{(0)} + \varepsilon\beta^{(1)} + \cdots,$$
$$R_\perp = R_\perp^{(0)} + \varepsilon R_\perp^{(1)} + \cdots, \quad H = H_0 + \varepsilon H_1^{(1)} + \cdots,$$
$$V_x = \varepsilon V_x^{(1)} + \cdots,$$

where the zeroth-order solution describes a stationary drop with a shape fully determined by surface tension, i.e., a spherical cap.

The first-order solution, in turn, describes small periodic perturbations of the drop's surface and small oscillations of the drop's position. Similar to the thin-drop case [6], two oscillatory modes will be distinguished: an axisymmetric mode describing periodic spreading and contraction of the drop and an asymmetric mode describing "swaying" of the drop up and down the slope. Following Ref. [5], we shall refer to the former and latter as the *spreading* and *swaying* modes, respectively.

Finally, the second-order solution describes the nonlinear interaction of the spreading and swaying modes, resulting in a mean drift of the drop up or down the slope.

A. The zeroth-order solution

The simplest expression for the leading-order solution can be obtained if one sets

$$H^{(0)} = -\cos\bar{\beta}. \tag{20}$$

Geometrically, this condition implies that the center of the spherical cap coincides with the origin.

Then, assuming that the leading-order solution is axisymmetric and time independent, one can readily find that

$$\begin{aligned} &R^{(0)} = 1, \quad \Theta^{(0)} = \bar{\beta}, \quad R_\perp^{(0)} = \sin\bar{\beta}, \\ &\beta^{(0)} = \bar{\beta}, \quad V_x^{(0)} = 0 \quad P^{(0)} = 2. \end{aligned} \tag{21}$$

B. The first-order solution

Expanding Eqs. (2)–(7), (12), (13), and (14), subject to conditions (16)–(18), to the first order and substituting, where necessary, the zeroth-order solution (20) and (21), we obtain

$$\begin{aligned} &2R^{(1)} + \frac{\cos\theta}{\sin\theta}\frac{\partial R^{(1)}}{\partial\theta} + \frac{\partial^2 R^{(1)}}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 R^{(1)}}{\partial\phi^2} \\ &\quad = -P^{(1)} + a^{(0)}(\sin\theta\cos\phi\sin\alpha + \cos\theta\cos\alpha), \end{aligned} \tag{22}$$

$$(R^{(1)})_{\theta=\bar{\beta}}\cos\bar{\beta} - \sin\bar{\beta}\,\Theta^{(1)} = -H^{(1)}, \tag{23}$$

$$R_\perp^{(1)} = -H^{(1)}\tan\bar{\beta} + \frac{\Theta^{(1)}}{\cos\bar{\beta}}, \tag{24}$$

$$\int_0^{2\pi}\Theta^{(1)}\cos\phi\,d\phi = 0, \tag{25}$$

$$\frac{\partial R_\perp^{(1)}}{\partial t} + V_x^{(1)}\cos\phi = \beta^{(1)}, \tag{26}$$

$$\beta^{(1)} = \Theta^{(1)} - \left(\frac{\partial R^{(1)}}{\partial\theta}\right)_{\theta=\bar{\beta}}, \tag{27}$$

$$\int_0^{2\pi}\int_0^{\bar{\beta}} R^{(1)}\sin\theta\,d\theta\,d\phi + \pi H^{(1)}\sin^2\bar{\beta} = 0. \tag{28}$$

We shall seek the solution of (22) in the form suggested by its right-hand side,

$$R^{(1)} = R_0^{(1)}(\theta,t) + R_1^{(1)}(\theta,t)\cos\phi, \tag{29}$$

where the first (axisymmetric) and the second (asymmetric) terms represent the spreading and swaying, respectively.

Substituting (29) into (22), we obtain the following equations for $R_0^{(1)}$ and $R_1^{(1)}$:

$$\frac{\partial^2 R_0^{(1)}}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial R_0^{(1)}}{\partial\theta} + 2R_0^{(1)} = a^{(0)}\cos\theta\sin\alpha - P^{(1)}, \tag{30}$$

$$\frac{\partial^2 R_1^{(1)}}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial R_1^{(1)}}{\partial\theta} + \left(2 - \frac{1}{\sin^2\theta}\right)R_1^{(1)} = a^{(0)}\sin\theta\sin\alpha. \tag{31}$$

Observe that the homogeneous versions of Eqs. (30) and (31) have particular solutions $\cos\theta$ and $\sin\theta$, respectively (which correspond to infinitesimal shifts of the drop in the directions perpendicular and parallel to the substrate, respectively). This circumstance allows one to find the general solution of (30) and (31). Requiring that $R_0^{(1)}$ and $R_1^{(1)}$ be finite at $\theta = 0$, one obtains

$$\begin{aligned} R_0^{(1)} &= \frac{a^{(0)}}{3}\left[\frac{1}{\cos\theta} - \ln(1+\cos\theta) - B(t)\right] \\ &\quad \times \cos\theta\cos\alpha - \frac{1}{2}P^{(1)}, \end{aligned} \tag{32}$$

$$R_1^{(1)} = \frac{a^{(0)}}{3}\left[\frac{1}{1+\cos\theta} - \ln(1+\cos\theta) - D(t)\right]\sin\theta\sin\alpha, \tag{33}$$

where the "constants of integration" $B(t)$ and $D(t)$ will be determined later.

Substitution of (29) into Eq. (23) yields

$$\Theta^{(1)} = \frac{H^{(1)}}{\sin\bar{\beta}} + \frac{\cos\bar{\beta}}{\sin\bar{\beta}}\left(R_0^{(1)} + R_1^{(1)}\cos\phi\right)_{\theta=\bar{\beta}}. \tag{34}$$

This expression can be simplified by taking advantage of the fact that $H^{(1)}(t)$ can be chosen at our discretion. We assume that

$$H^{(1)} = -\left(R_0^{(1)}\right)_{\theta=\bar{\beta}}\cos\bar{\beta}, \tag{35}$$

after which (34) becomes

$$\Theta^{(1)} = \frac{\cos\bar{\beta}}{\sin\bar{\beta}}\left(R_1^{(1)}\right)_{\theta=\bar{\beta}}\cos\phi. \tag{36}$$

Substitution of this expression into condition (25) yields

$$\left(R_1^{(1)}\right)_{\theta=\bar{\beta}} = 0, \tag{37}$$

then (36) yields

$$\Theta^{(1)} = 0. \tag{38}$$

Note that, if we fixed the position of the origin (and, thus, H) from the start, we would not be able to make $\Theta^{(1)}$ zero. As a result, all of the following calculations would be much more cumbersome.

Condition (37) also fixes the constant of integration in expression (33) for $R_1^{(1)}$,

$$D = \frac{1}{1+\cos\bar{\beta}} - \ln(1+\cos\bar{\beta}). \tag{39}$$

Finding the rest of the first-order solution is straightforward, so the details will be omitted; we only mention that the unknowns can be expressed through B [which appears in (32)]. In what follows, we shall need

$$P^{(1)} = \frac{a^{(0)}\cos\alpha}{3}\,\frac{\frac{3}{2} + \cos\bar{\beta} - (1+\cos\bar{\beta})^2\ln(1+\cos\bar{\beta}) - (1+\cos\bar{\beta})^2 B}{1 + \frac{1}{2}\cos\bar{\beta}}, \tag{40}$$

and the local contact angle,

$$\beta^{(1)} = \beta_0^{(1)}(t) + \beta_1^{(1)}(t)\cos\phi, \tag{41}$$

where

$$\beta_0^{(1)} = -\frac{a^{(0)}\sin\bar{\beta}\cos\alpha}{3}\left[\frac{\cos\bar{\beta}}{1+\cos\bar{\beta}} + \ln(1+\cos\bar{\beta}) + B\right], \quad \beta_1^{(1)} = -\frac{a^{(0)}\sin\alpha}{3}\frac{2-\cos\bar{\beta}-\cos^2\bar{\beta}}{1+\cos\bar{\beta}}. \tag{42}$$

The function $B(t)$, in turn, is related to the acceleration of the substrate, $a^{(0)}(t)$, by an ordinary differential equation,

$$\frac{1}{2+\cos\bar{\beta}}\frac{d(a^{(0)}B)}{dt} + (a^{(0)}B) = -\frac{\frac{1}{2}+\ln(1+\cos\bar{\beta})}{2+\cos\bar{\beta}}\frac{da^{(0)}}{dt} - \left[\frac{\cos\bar{\beta}}{1+\cos\bar{\beta}} + \ln(1+\cos\bar{\beta})\right]a^{(0)}. \tag{43}$$

We shall also present the expression for the velocity,

$$V_x^{(1)} = -\frac{a^{(0)}\sin\alpha}{3}\frac{2-\cos\bar{\beta}-\cos^2\bar{\beta}}{1+\cos\bar{\beta}},$$

which shows that $V_x^{(1)}(t)$ is periodic with *zero mean* [because the substrate's acceleration $a^{(0)}(t)$ is periodic with zero mean]. Thus, to find the mean rise velocity of the drop, one needs to examine the next order.

To illustrate the zeroth-order and first-order solutions, we used them to plot the drop's shape and its relative displacement for different phases of the oscillation (see Fig. 2). The swaying mode is clearly visible in this figure (especially when $t = \pi/2, 3\pi/2$), whereas the spreading mode is harder to spot: it is responsible for widening or narrowing of the drop.

C. The second-order solution

The calculations associated with the second order are straightforward, but tedious, and are thus presented in the Appendix. In the main body of the paper, we shall state the main result only, namely, the expression for the rise velocity,

$$V_x^{(2)} = \left\{\frac{\cos\alpha}{3}\left[\frac{(1-\cos\bar{\beta})a^{(0)2}}{2(2+\cos\bar{\beta})(1+\cos\bar{\beta})} + \left(\frac{1}{2+\cos\bar{\beta}} - \cos\bar{\beta} + \mu\sin\bar{\beta}\right)a^{(0)}A\right] - \frac{a^{(1)}\sin\alpha}{\varepsilon}\right\} \times \frac{2-\cos\bar{\beta}-\cos^2\bar{\beta}}{3(1+\cos\bar{\beta})}\sin\alpha, \tag{44}$$

where the function $A(t)$ satisfies

$$\frac{dA}{dt} + (2+\cos\bar{\beta})A = -\frac{1-\cos\bar{\beta}}{2(1+\cos\bar{\beta})}\frac{da^{(0)}}{dt}. \tag{45}$$

Equations (44) and (45) determine the dependence of $V_x^{(2)}$ on the equilibrium contact angle $\bar{\beta}$, the nondimensional gravity $a^{(1)}$, the substrate's acceleration $a^{(0)}$, and its inclination angle α.

IV. AN EXAMPLE: THE CASE OF SINUSOIDAL VIBRATION

Let the substrate's acceleration be

$$a^{(0)}(t) = \sin\omega t, \tag{46}$$

which corresponds to the experiments [1,2]. Substituting (46) into Eq. (45) and solving it, we obtain

$$A = -\frac{\omega[\omega\sin\omega t + (2+\cos\bar{\beta})\cos\omega t]}{2[\omega^2 + (2+\cos\bar{\beta})^2]}\frac{1-\cos\bar{\beta}}{1+\cos\bar{\beta}}. \tag{47}$$

We shall only need the mean of $V_x^{(2)}(t)$, given by

$$\langle V_x^{(2)}\rangle = \frac{1}{T}\int_0^T V_x^{(2)}dx, \tag{48}$$

where $T = 2\pi/\omega$ is the period of the vibration. Substituting (46) and (47) into (44), and (44) into (48), we obtain

$$\langle V_x^{(2)}\rangle = \left\{\frac{(1-\cos\bar{\beta})\cos\alpha}{12(1+\cos\bar{\beta})}\left[\frac{1}{2+\cos\bar{\beta}} - \left(\frac{1}{2+\cos\bar{\beta}} - \cos\bar{\beta} + \mu\sin\bar{\beta}\right)\frac{\omega^2}{\omega^2+(2+\cos\bar{\beta})^2}\right] - \frac{a^{(1)}}{\varepsilon}\right\} \times \frac{2-\cos\bar{\beta}-\cos^2\bar{\beta}}{3(1+\cos\bar{\beta})}\sin\alpha. \tag{49}$$

It can be extracted from this expression that $\langle V_x^{(2)}\rangle$ is positive (i.e., the drop climbs uphill) if and only if

$$\varepsilon_* > \frac{12(1+\cos\bar{\beta})[(2+\cos\bar{\beta})^2+\omega^2]}{(1-\cos\bar{\beta})[2+\cos\bar{\beta}+(\cos\bar{\beta}-\eta\sin\bar{\beta})\omega^2]}, \tag{50}$$

where

$$\varepsilon_* = \frac{\varepsilon\cos\alpha}{a^{(1)}}.$$

Criterion (50) is the main result of this work.

To understand its physical meaning, we shall use (15) to express ε_* through the dimensional variables,

$$\varepsilon_* = \frac{\rho a_0^2 R_0^2\cos\alpha}{\sigma g}. \tag{51}$$

Now one can see that ε_* characterizes the strength of the vibration relative to gravity and surface tension. The relationship of the nondimensional frequency ω of the vibration to its dimensional counterpart ω_{dim} can be extracted from (10),

$$\omega = \frac{R_0}{v'}\omega_{\mathrm{dim}}, \tag{52}$$

where it should be recalled that v' is the derivative of the contact-line law $v(\beta)$ at the equilibrium value of the contact angle, $\beta = \bar{\beta}$. Criterion (50) is illustrated on the (ω, ε_*) plane in Fig. 3.

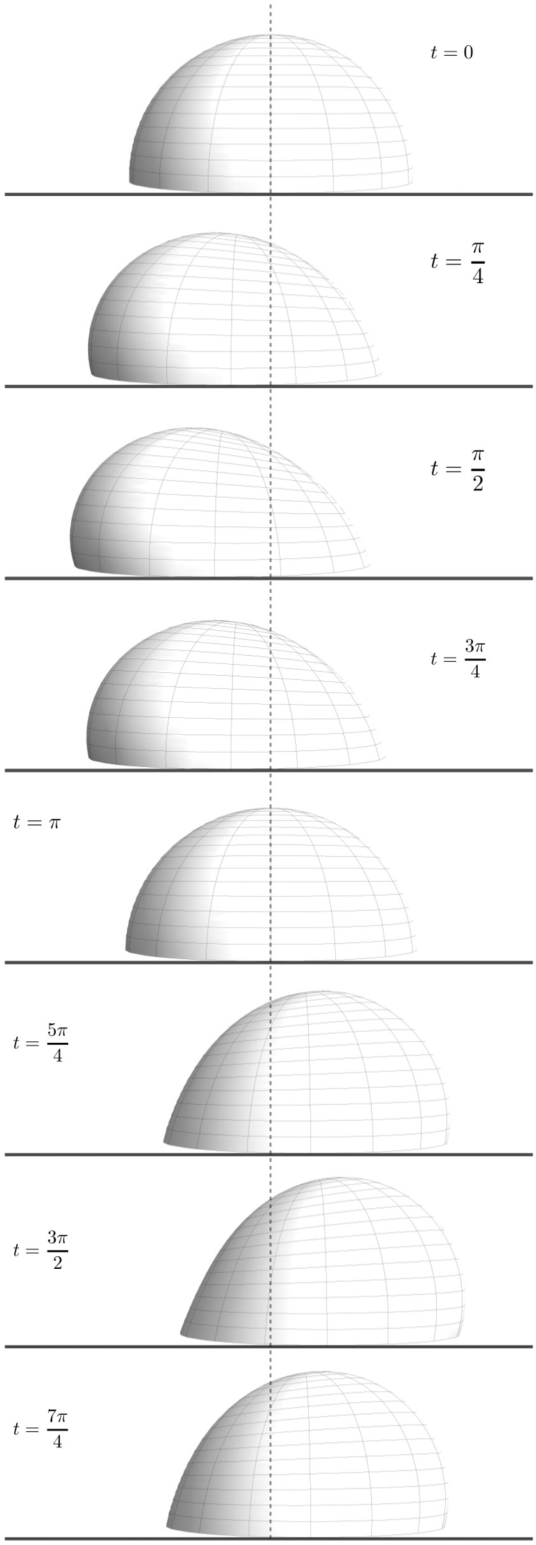

FIG. 2. The drop's shape and relative displacement for different phases of the oscillation, for $\bar{\beta} = 90^\circ$, $\alpha = 45^\circ$, $\omega = 1$, and $\varepsilon = 0.5$ (such a large value of ε was used to make the dynamics more "visible").

V. DISCUSSION

(1) Observe that, if the substrate is horizontal, gravity and the vibration-induced inertial force are perpendicular to it and, thus, cannot move the drop either left or right. Accordingly, expression (49) for the rise velocity vanishes for $\alpha = 0$.

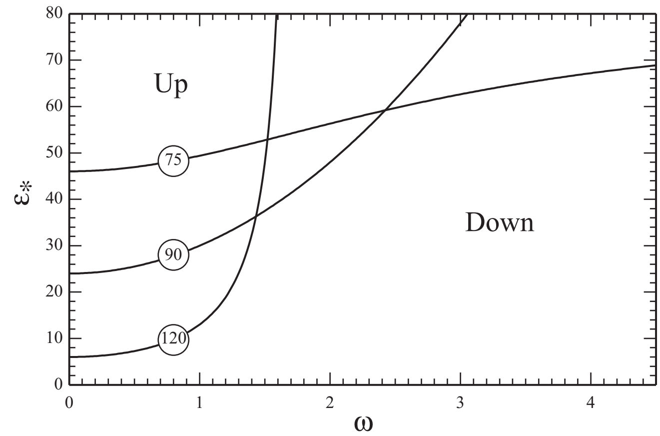

FIG. 3. Criterion (50) on the (ω,ε_*) plane, for the particular case $\mu = 0$. The parameters ε_* and ω are given by (51) and (52). The curves are labeled with the corresponding values of the equilibrium contact angle $\bar{\beta}$. The parameter region above, and to the left from, a curve corresponds to the drop with this value of $\bar{\beta}$ climbing uphill. If $\bar{\beta}$ satisfies condition (53) (as $\bar{\beta} = 120^\circ$ does), the corresponding curve has a vertical asymptote at a finite value of ω.

It is less obvious why the vibration-induced part of the rise velocity [i.e., the first term in the curly braces in expression (49)] vanishes if $\alpha = 90^\circ$. To clarify this issue, recall that the drop's uphill motion is caused by the interaction of the spreading and swaying oscillatory modes generated by the vibration. However, if the substrate is vertical, the spreading mode is *not* excited, and the swaying mode alone cannot make the drop climb uphill.

(2) Figure 3 suggests that, for sufficiently large values of the equilibrium contact angle $\bar{\beta}$, the region where the drop climbs uphill is bounded on the right by a vertical asymptote. As follows from (50), this occurs when

$$\cot \bar{\beta} < \eta, \tag{53}$$

where η is defined by (15). It also follows from (50) that, if condition (53) holds, the vertical asymptote is located at

$$\omega_{cr} = \sqrt{\frac{2 + \cos \bar{\beta}}{\eta \sin \bar{\beta} - \cos \bar{\beta}}}.$$

To understand why ε_* becomes infinite as $\omega \to \omega_{cr}$, observe that the vibration-induced part of the rise velocity vanishes at $\omega = \omega_{cr}$ (see an example in Fig. 4). As a result, if ω approaches ω_{cr} from the left, the vibration must be increasingly strong to make the drop climb uphill. Once ω passes ω_{cr}, the vibration-induced velocity becomes negative, and, no matter how strong the vibration is, the drop slides downhill.

Note also that, even thought the mean vibration-induced velocity vanishes at $\omega = \omega_{cr}$, its nonaveraged counterpart does not [see (44) with $a^{(1)} = 0$]. This shows that neither swaying, nor spreading mode vanishes at $\omega = \omega_{cr}$, but their interaction has zero mean effect on the drop.

(3) In the limit $\bar{\beta} \ll 1$, criterion (50) coincides with its thin-drop counterpart, Eqs. (51) of Ref. [6] (provided a typo is corrected in the latter [14]).

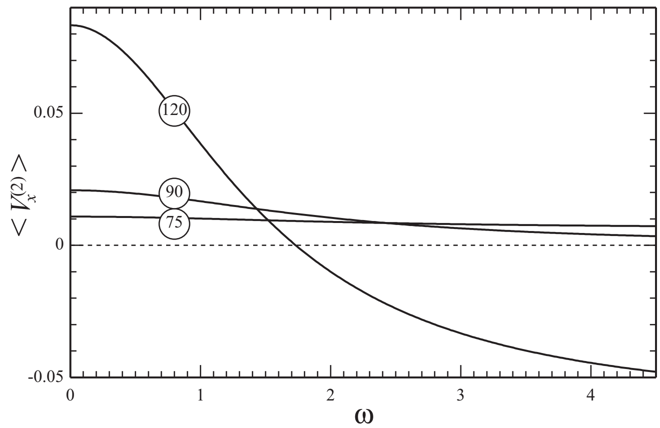

FIG. 4. The vibration-induced part of the rise velocity [given by (49) with $a^{(1)} = 0$] for $\alpha = 45^\circ$. The curves are labeled with the corresponding values of the equilibrium contact angle $\bar{\beta}$. The point where $\langle V_x^{(2)} \rangle = 0$ for $\bar{\beta} = 120^\circ$ corresponds to the vertical asymptote of the curve for the same value of $\bar{\beta}$ in Fig. 3.

The thin-drop case, however, is of limited practical importance, as it is extremely difficult to make a thin drop climb uphill [7]. This experimental observation agrees with the fact that, for small $\bar{\beta}$, criterion (50) yields very high values of ε_* (see Fig. 3).

(4) A further reduction in ε_* occurs when the equilibrium contact angle $\bar{\beta}$ tends to 180°, i.e. when the substrate is hydrophobic (this tendency is also visible in Fig. 3).

(5) Observe that expression (49) for the rise velocity, as well as criterion (50), both involve the parameter η [given by (15)], which, in turn, depends on the derivatives v' and v'' of the contact-line law, $v(\beta)$, at $\beta = \bar{\beta}$. Thus, unless v' and v'' are known, our theoretical results cannot be compared with an experiment.

There are two ways to determine v' and v''. In principle, one can try to measure $v(\beta)$ experimentally, in a similar fashion to Ref. [15], and then calculate v' and v''. It is unlikely, however, that an experimentally measured $v(\beta)$ can be accurate enough for calculating its first and (even more so) second derivatives.

Alternatively, one can determine v' and v'' directly from an experiment with drops on a vibrating inclined substrate. With this approach, two calibrating measurements of the rise velocity should be made for two different values of the frequency ω. The measured values should then be equated to the theoretical expression, resulting in a set of two equations for v' and v''. Once v' and v'' are determined, they can be used for all other frequencies and amplitudes of the vibrations, and all slopes of the substrate.

(6) Finally, we discuss whether the quasistatic and small-ε approximations apply to the existing experiments. As shown in Ref. [9], the former approximation holds if

$$\lambda = \frac{\rho\omega^2 R_0^3}{\sigma} \ll 1, \quad \mu = \frac{\rho\omega\nu R_0}{\sigma} \ll 1,$$

where ν is the kinematic viscosity and the rest of the notation was explained earlier. Note that λ and μ characterize the inertial and viscous forces, relative to surface tension.

We have calculated λ, μ, and ε for the experiments represented in Fig. 2 of Ref. [1]. These experiments were carried out for a mixture of water and glycerine, such that

$$\rho = 1190\ \mathrm{kg\,m^{-3}}, \quad \sigma = 0.066\ \mathrm{N\,m^{-1}}, \quad \nu = 0.000\,031\ \mathrm{m^2\,s^{-1}}, \tag{54}$$

and the slope of the substrate was

$$\alpha = 45^\circ. \tag{55}$$

Drops climbing uphill were observed when the frequency of the substrate's vibration and its acceleration varied in the ranges

$$\omega \approx 30\text{–}120\ \mathrm{Hz}, \quad a_0 \approx 12.2g\text{–}25.7g. \tag{56}$$

Calculating λ and μ for parameters (54)–(56), we obtain

$$\lambda \approx 0.08\text{–}1.28, \quad \mu = 0.03\text{–}0.11.$$

Thus, the quasistatic approximation did hold for some of the experiments of Ref. [1].

Next, substituting (54)–(56) in definition (15) of ε, we obtain

$$\varepsilon \approx 6.3\text{–}13.3.$$

Thus, ε was *not* small in the experiments of Ref. [1].

To understand why ε turned out to be relatively large, recall that Ref. [1] used a liquid-substrate combination with the receding and advancing contact angles of $\beta_r = 44^\circ$ and $\beta_a = 77^\circ$. Assuming that the hysteresis interval (β_r, β_a) can be qualitatively approximated by a single equilibrium angle halfway between β_r and β_a, i.e., $\bar{\beta} = 60.5^\circ$, our comment (4) suggests that the drops in this case can be made to climb uphill only by a very strong vibration. To reduce its strength, one has to use a *hydrophobic* substrate, such that the center of the hysteresis interval is as close as possible to 180°.

VI. CONCLUDING REMARKS

Thus, we have examined the evolution of drops on an inclined substrate oscillating vertically, and our main result is criterion (50). The analysis is based on the following three assumptions. A1: The frequency of the vibration is sufficiently low, so the quasistatic approximation can be used. A2: The substrate's oscillations are weak [so that the parameter ε, given by (15), is small]. A3: The contact-line law does not involve a hysteresis interval.

The restriction to *thin* drops, used in all previous theoretical studies [4–6], has now been relaxed, which makes it possible to test the present theory against a specifically designed experiment. Indeed, assumptions A1 and A2 can be readily satisfied by choosing an appropriate frequency and amplitude of the substrate's vibration [16]. Assumption A3, in turn, can be satisfied by using liquid-substrate combinations with narrow hysteresis intervals (such as those examined in Refs. [12,13]).

Note also that assumption A3 is the most restrictive of the three, as most liquid-substrate combinations have contact-line laws with sizable hysteresis intervals. In such cases our asymptotic analysis is inapplicable, but our governing equations still are, and, in principle, can be integrated numerically. The main difficulty associated with this approach is solving Eq. (1) (which is a nonlinear elliptic equation) in a region of varying shape.

Finally, note that all of the experimental and theoretical results cited above were obtained for harmonic vibration of the substrate. The case of *anharmonic* vibration has also been examined, both experimentally [17,18] and theoretically [9]. It appears that an anharmonic vibration affects the drops stronger than a sinusoidal one. However, before a comparison of theoretical and experimental results for this case is possible, one needs to extend the theoretical results of Ref. [9] to three spatial dimensions.

ACKNOWLEDGMENT

The authors acknowledge the support of the Science Foundation Ireland under Grants No. 11/RFP.1/MTH3281 and No. 12/IA/1683.

APPENDIX: THE SECOND-ORDER SOLUTION

One can readily deduce that the second-order solution has the form

$$R^{(2)}(\theta,\phi,t) = R_0^{(2)}(\theta,t) + R_1^{(2)}(\theta,t)\cos\phi + R_2^{(2)}(\theta,t)\cos 2\phi. \tag{A1}$$

It turns out, however, that the rise velocity $V_x^{(2)}$ depends only on $R_1^{(2)}$, and the boundary-value problem for $R_1^{(2)}$ does *not* involve $R_0^{(2)}$ and $R_2^{(2)}$. Thus, we shall keep track only of $R_1^{(2)}$.

Similarly to (A1), we let

$$\Theta^{(2)}(\phi,t) = \Theta_0^{(2)}(t) + \Theta_1^{(2)}(t)\cos\phi + \Theta_2^{(2)}(t)\cos 2\phi,$$
$$R_\perp^{(2)}(\phi,t) = R_{\perp 0}^{(2)}(t) + R_{\perp 1}^{(2)}(t)\cos\phi + R_{\perp 2}^{(2)}(t)\cos 2\phi,$$
$$\beta^{(2)}(\phi,t) = \beta_0^{(2)}(t) + \beta_1^{(2)}(t)\cos\phi + \beta_2^{(2)}(t)\cos 2\phi,$$

where we need to keep track only of $\Theta_1^{(2)}$, $R_{\perp 1}^{(2)}$, and $\beta_1^{(2)}$.

The second-order Eqs. (5) and (4) yield $\Theta_1^{(2)} = 0$ and $R_{\perp 1}^{(2)} = 0$. Substituting expressions (29) and (41) for $R^{(1)}$ and $\beta^{(2)}$ in the second order of (6) and (7), we obtain

$$V_x^{(2)} = \beta_1^{(2)} + \eta\beta_0^{(1)}\beta_1^{(1)},$$
$$\beta_1^{(2)} = \beta_0^{(1)}\beta_1^{(1)}\tan\bar\beta + \left(R_0^{(1)}\frac{\partial R_1^{(1)}}{\partial\theta} + R_1^{(1)}\frac{\partial R_0^{(1)}}{\partial\theta} - \frac{\partial R_0^{(1)}}{\partial\theta}\frac{\partial R_1^{(1)}}{\partial\theta}\tan\bar\beta - \frac{\partial R_1^{(2)}}{\partial\theta}\right)_{\theta=\bar\beta},$$

which can be rearranged as a boundary condition for $R_1^{(2)}$,

$$\left(\frac{\partial R_1^{(2)}}{\partial\theta}\right)_{\theta=\bar\beta} = F - V_x^{(2)}, \tag{A2}$$

where

$$F = (\eta + \tan\bar\beta)\beta_0^{(1)}\beta_1^{(1)} + \left(R_0^{(1)}\frac{\partial R_1^{(1)}}{\partial\theta} + R_1^{(1)}\frac{\partial R_0^{(1)}}{\partial\theta} - \frac{\partial R_0^{(1)}}{\partial\theta}\frac{\partial R_1^{(1)}}{\partial\theta}\tan\bar\beta\right)_{\theta=\bar\beta}. \tag{A3}$$

The second-order Eqs. (2), (12), and (18) can be reduced to a single equation, which can be simplified using (30) and (31) and becomes

$$\frac{\partial^2 R_1^{(2)}}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial R_1^{(2)}}{\partial\theta} + \left(2 - \frac{1}{\sin^2\theta}\right)R_1^{(2)} = G, \tag{A4}$$

where

$$G = -2R_1^{(1)}\big(P^{(1)} + 2R_0^{(1)}\big)\sin\alpha + 3a^{(0)}\big(R_0^{(1)}\sin\theta\sin\alpha + R_1^{(1)}\cos\theta\cos\alpha\big) + a^{(1)}\sin\theta\sin\alpha. \tag{A5}$$

Next, it follows from (3) that

$$\big(R_1^{(2)}\big)_{\theta=\bar\beta} = 0. \tag{A6}$$

Finally, we impose on $R_1^{(2)}$ a condition of regularity at $\theta = 0$ [which is a singular point of Eq. (A4)], i.e.,

$$\big(\big|R_1^{(2)}\big|\big)_{\theta=0} < \infty. \tag{A7}$$

Since the boundary-value problem (A2), (A4), (A6), and (A7) includes a *second*-order equation and *three* boundary conditions, it appears to be overdetermined. However, the right-hand side of the boundary condition (A2) involves an unknown quantity, $V_x^{(2)}$, which can be adjusted to satisfy the extra boundary condition.

In fact, one can find $V_x^{(2)}$ without solving the boundary-value problem (A2), (A4), (A6), and (A7). Indeed, multiply Eq. (A4) by $\sin^2\theta$ and integrate it with respect to θ, from 0 to $\bar\beta$. Integrating the term involving $\partial^2 R_1^{(2)}/\partial\theta^2$ by parts twice and taking into account the boundary conditions (A2), (A6), and (A7), one obtains

$$V_x^{(2)} = F - \frac{1}{\sin^2\bar\beta}\int_0^{\bar\beta} G\sin^2\theta\, d\theta. \tag{A8}$$

Given that expressions (A3) and (A5) for F and G involve only the *first*-order unknowns (for which explicit expressions are presented in Sec. III B), (A8) is essentially an expression for the rise velocity $V_x^{(2)}$.

To obtain a self-contained expression for $V_x^{(2)}$, one needs to substitute $R_0^{(1)}$ [given by (32) and (40)], $R_1^{(1)}$ [given by (33) and (39)], $\beta_0^{(1)}$ and $\beta_1^{(1)}$ [given by (42)] into expressions (A3) and (A5) for F and G. Then, substituting F and G into expression (A8), we obtain [19] expression (44), where the function $A(t)$ is related to $B(t)$ by

$$A(t) = \left[B(t) + \frac{\cos\bar\beta}{1+\cos\bar\beta} + \ln(1+\cos\bar\beta)\right]a^{(0)}. \tag{A9}$$

It can be deduced from (43) that A satisfies Eq. (45).

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