

Short-wave, localized disturbances in jets, with applications to flows on a beta plane with topography

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We examine the stability of jets over topography on the so-called barotropic beta plane (which models oceanic currents in mid-latitudes). Attention is focused on disturbances with a large wave number, for which an asymptotic solution of the normal-mode eigenvalue problem is presented. It is demonstrated that short-wave modes, if they exist, are localized in narrow strips “centered” at local extrema of the velocity profile $U(y)$ (y is the transverse variable). It is further shown that an extremum, say $y=y_l$, can support a short-wave mode only if the vorticity gradient and potential vorticity (PV) gradient at y_l are of opposite signs. If they are, the leading-order solution of the eigenvalue problem describes a mode with a phase speed slightly larger than $U(y_l)$, if y_l is a maximum; or slightly smaller than $U(y_l)$, if y_l is a minimum. In other words, the mode does not have critical levels in the vicinity of y_l , although it may have them elsewhere, at a distant point. If that indeed happens, an additional condition is required to guarantee the existence of the mode: namely, the PV gradient at y_l and that at the critical level must have opposite signs. If they do, the mode exists and is weakly unstable (the phase speed has a small imaginary part). Thus, a change of sign of the PV gradient does not necessarily destabilize the flow; and in order to guarantee instability, the PV gradient should have opposite signs at the “important” points, i.e., the localization point and critical level. The asymptotic results are tested against the numerical solution of the exact normal-mode eigenvalue problem. The former and the latter are in good agreement, and not only for large wave numbers, but also for moderate ones. It is also demonstrated that our approach can be applied to other cases of jets in fluids and plasmas. © 2003 American Institute of Physics. [DOI: 10.1063/1.1540100]

I. INTRODUCTION

A large number of papers is published on hydrodynamic stability every year. Most of them examine various examples of stable or unstable flows, with a relatively small proportion devoted to the techniques used in this branch of fluid dynamics. The present paper is of the latter kind, with its results being applicable to any system that is unstable with respect to harmonic disturbances.

It has originated from the observation that, in some cases, hydrodynamic instability has a short-wave cutoff, i.e., disturbances with sufficiently short wavelengths are stable (or, more often than not, simply do not exist as solutions of the corresponding eigenvalue problem). In other cases, the spectral range of instability goes all the way to infinity. What makes those cases different? Is there a way to calculate the parameters of short-wave instability when it occurs? These questions are answered in the present paper for the particular case of inviscid two-dimensional homogeneous fluid.

As an illustration of the general technique, we consider a relatively simple, but oceanographically important, example of barotropic currents over an uneven bottom, on the beta plane. In Sec. II, we formulate the problem and derive an equation for linear harmonic disturbances (normal modes). This equation is examined in the short-wave limit in Sec. III. No unstable modes are found within the framework of the leading-order asymptotic analysis. It turns out, however, that

the leading-order analysis ignores the singularities associated with *critical levels*, and in Secs. IV and V, the effect of those is examined. In Sec. VI, we verify the asymptotic results by comparison with the exact solution computed numerically, and in Sec. VII, outline how our approach can be applied to other flows in fluid and plasma.

II. FORMULATION OF THE PROBLEM

Consider a barotropic ocean of variable depth $H(x,y)$, where x and y are the horizontal coordinates (the x axis is directed eastward), on the beta plane. The flow is characterized by the streamfunction $\psi(t,x,y)$, where t is the time variable. We assume ψ to be governed by the barotropic vorticity equation (see Ref. 1),

$$\frac{\partial \nabla^2 \psi}{\partial t} + J \left(\psi, \nabla^2 \psi - \frac{f_0}{H_0} H \right) + \beta \frac{\partial \psi}{\partial x} = 0, \quad (1)$$

where J is the Jacobian, H_0 is the average $H(x,y)$, and f_0 and β are the Coriolis parameter and its meridional gradient, respectively.

Let $U(y)$ describe a zonal flow, aligned with a zonal topography $H(y)$ —see Fig. 1. Assuming that the flow is perturbed by a small disturbance, $\tilde{\psi}$, we set

$$\psi = - \int U(y) dy + \tilde{\psi}. \quad (2)$$

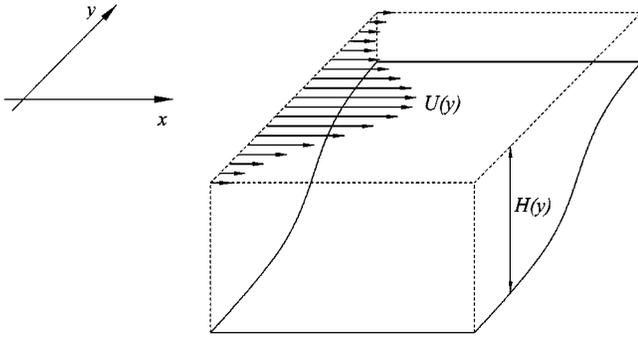


FIG. 1. Formulation of the problem: a zonal jet over zonal topography.

Substituting (2) into (1), and omitting nonlinear terms, we obtain (over tildes dropped)

$$\frac{\partial \nabla^2 \psi}{\partial t} + U \frac{\partial \nabla^2 \psi}{\partial x} + \left(\beta - \frac{f_0}{H_0} \frac{dH}{dy} - \frac{d^2 U}{dy^2} \right) \frac{\partial \psi}{\partial x} = 0. \quad (3)$$

This paper is devoted to normal modes, i.e., solutions of the form

$$\psi(x, y, t) = \text{Re}[\phi(y)e^{ik(x-ct)}], \quad (4)$$

where k and c are the wave number and phase speed. Substituting (4) into (3), we obtain

$$\frac{d^2 \phi}{dy^2} - k^2 \phi + \frac{Q'}{U-c} \phi = 0, \quad (5)$$

where

$$Q' = \beta - \frac{f_0}{H_0} \frac{dH}{dy} - \frac{d^2 U}{dy^2} \quad (6)$$

is, physically, the potential vorticity (PV) gradient. Equation (5) should be supplemented by the usual boundary conditions,

$$\phi \rightarrow 0, \quad \text{as } y \rightarrow \pm \infty. \quad (7)$$

Here (5) and (7) form an eigenvalue problem, where ϕ is the eigenfunction and c is the eigenvalue. If $\text{Im } c > 0$, the flow is unstable.

III. SHORT-WAVE, LOCALIZED DISTURBANCES

We shall nondimensionalize the problem using the amplitude U_0 and spatial scale Y of the flow,

$$y_* = \frac{y}{Y}, \quad U_* = \frac{U}{U_0}, \quad H_* = \frac{f_0 Y H}{U_0 H_0}, \quad k_* = Yk,$$

$$c_* = \frac{c}{U_0}.$$

Rewriting (5) and (6) in terms of the new variables and omitting the asterisk, one can see that (5) remains the same, whereas (6) becomes

$$Q' = \alpha - \frac{dH}{dy} - \frac{d^2 U}{dy^2}, \quad (8)$$

where

$$\alpha = \frac{\beta Y^2}{U_0}.$$

The present paper is devoted to the limit

$$k \rightarrow \infty,$$

i.e., dimensionally, we assume that the wavelength of the disturbance is much smaller than the width of the jet. Equation (5), to the leading order, becomes

$$\frac{d^2 \phi}{dy^2} - k^2 \phi \approx 0, \quad (9)$$

which clearly has no bounded solutions. Observe, however, that (9) is not applicable near the points where the denominator of the third term in (5) is small, making it comparable to k^2 . Thus, we conclude that the eigenfunction is localized near one of such points, if any. It is unclear, however, where exactly the localization point is situated.

Assume, for example, that ϕ is localized near a “generic” point, say y_l , where $U(y)$ does *not* have an extremum,

$$\left(\frac{dU}{dy} \right)_{y=y_l} \neq 0.$$

Then, in the spirit of the approximation developed in Refs. 2–4 we expand the coefficients of Eq. (5) near the localization point,

$$\frac{Q'}{U-c} = \frac{Q'_l + \dots}{U_l + U'_l(y-y_l) + \dots - c},$$

where

$$U_l = U(y_l), \quad U'_l = \left(\frac{dU}{dy} \right)_{y=y_l}, \quad Q'_l = Q'(y_l).$$

At the localization point, $U-c$ should be small, which is possible only if the leading-order term of the expansion of U is cancelled by c , i.e., $c \approx U_l$. Hence, to the leading order, Eq. (5) can be approximated by

$$\frac{d^2 \phi}{dy^2} - k^2 \phi + \frac{Q'_l}{U_l + U'_l(y-y_l) - c} \phi \approx 0. \quad (10)$$

This equation describes the structure of the eigenfunction (if that exists) near the localization point. It turns out, however, that the eigenvalue problem (10), (7) has no solutions (see Appendix A). We conclude that ϕ may not be localized near a “generic” point.

We shall now assume that y_l is an extremum (but not, for simplicity, an inflection point), i.e.,

$$U'_l = 0, \quad U''_l \neq 0.$$

Keeping in mind, as before, that $c \approx U_l$, we expand Eq. (5) about y_l and, to the leading order, obtain

$$\frac{d^2 \phi}{dy^2} - k^2 \phi + \frac{Q'_l}{U_l + \frac{1}{2} U''_l (y - y_l)^2 - c} \phi \approx 0. \quad (11)$$

This equation can be reduced to an ‘‘oblate spheroidal wave equation’’ (see Ref. 5). Unfortunately, nothing seems to be known about its solutions, in spite of its having a name, so we must examine them ourselves.

It can be demonstrated (see Appendix B) that the eigenvalue problem (11), (7) may not have complex (unstable) eigenvalues. It can further be proven that, if

$$\frac{Q'_l}{U''_l} > 0, \quad (12)$$

there exists at least one real eigenvalue. On the other hand, if (12) does *not* hold, (11), (7) do not have solutions at all. Thus, not *any* extremum of U can support a localized mode, but only such that its parameters satisfy condition (12). Observe also that (12) can be interpreted as a requirement that, at the localization point, the vorticity gradient $-U''_l$ and the PV gradient Q'_l be of opposite signs.

Finally, we shall present the following estimate for the eigenvalues (if those exist):

$$\frac{c - U_l}{U''_l} < 0 \quad (13)$$

(see Appendix B). Inequality (13) is extremely important, as it prevents the denominator in the ‘‘local’’ equation (11) from vanishing, i.e. having a critical level (this question will be addressed in more detail in the next section).

We shall not dwell on the ‘‘local’’ eigenvalue problem (11), (7) in more detail, but mention only that, due to the regularity of its coefficients, it can be readily solved numerically. We also present some asymptotic solutions in Appendix C.

IV. THE EFFECT OF CRITICAL LEVELS

As mentioned above, condition (13) guarantees that the coefficients of the ‘‘local’’ equation (11) are regular for all y . In terms of the exact equation (5), however, this applies only to the vicinity of y_l and does not prevent $U - c$ from vanishing elsewhere (see Fig. 2). In this case, the coefficients of (5) are singular, which may ‘‘eliminate’’ all solutions (make them nonexistent). Alternatively, the exact eigenvalue may have a small imaginary part, which has been missed by the (leading-order) ‘‘local’’ equation. An imaginary part, no matter how small, would not allow $U - c$ be *exactly* zero.

Thus, there are two possibilities: (i) the exact equation (5) may not have solutions corresponding to the solutions of the asymptotic equation (11); and (ii) the exact eigenvalues may have small imaginary parts. In this section, we shall find out which of the possibilities is true.

We shall start from an identity, which can be derived via multiplying the exact equation (5) by ϕ^* and integrating

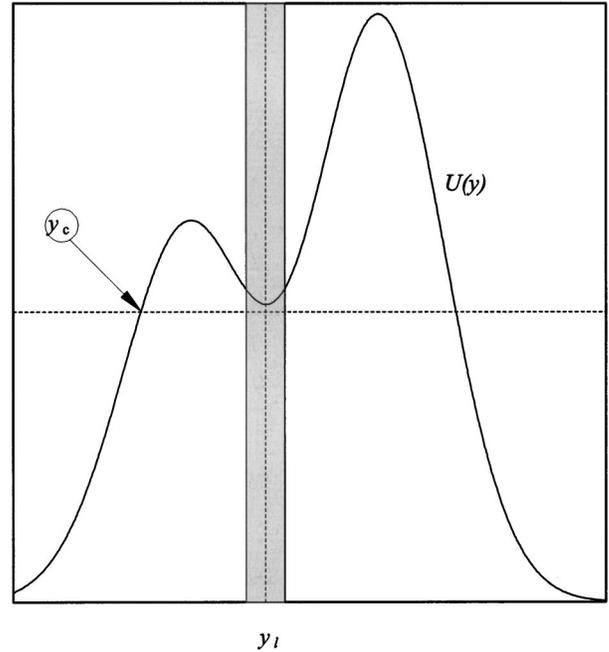


FIG. 2. A localized mode with a critical level. y_l is the localization point; y_c is the critical level closest to y_l . The localization region is shaded. Short-wave modes can also be localized near the two maxima of $U(y)$.

over $-\infty < y < \infty$. Integrating the term $\phi_{yy} \phi^*$ by parts, using the boundary condition (7), and taking the imaginary part of the resulting equality, we obtain

$$\int_{-\infty}^{\infty} \frac{c_i Q'_l}{(U - c_r)^2 + c_i^2} |\phi|^2 dy = 0, \quad (14)$$

where $c_r = \text{Re } c$, $c_i = \text{Im } c$.

We assume that the mode has a *critical level*, i.e., a point y_c such that

$$U(y_c) = c_r.$$

We shall further assume that the localization point and critical level are wide apart,

$$|y_l - y_c| \gg r,$$

where r is the ‘‘localization radius’’ of the eigenfunction (it can be readily shown that $r \sim k^{-1}$, hence, r is small, and the above condition is not very restrictive). The fact that the asymptotic eigenvalues [solutions of the ‘‘local’’ problem (11), (7)] are real suggests that the imaginary parts of the exact eigenvalues are small,

$$|c_r| \gg |c_i|$$

(it will be shown later that c_i is, in fact, *exponentially* small). Then, (14) can be simplified by subdividing the region of integration into a ‘‘negative periphery’’ ($y < y_l - r$), a ‘‘core’’ ($y_l - r < y < y_l + r$), and a ‘‘positive periphery’’ ($y > y_l + r$):

$$I_- + I_0 + I_+ = 0, \quad (15)$$

where

$$I_- = \int_{-\infty}^{y_l-r} \frac{c_i Q'}{(U-c_r)^2 + c_i^2} |\phi|^2 dy,$$

$$I_+ = \int_{y_l+r}^{\infty} \frac{c_i Q'}{(U-c_r)^2 + c_i^2} |\phi|^2 dy,$$

$$I_0 = \int_{y_l-r}^{y_l+r} \frac{c_i Q'}{(U-c_r)^2 + c_i^2} |\phi|^2 dy.$$

Within the core, we can use the “local” approximation. Expanding, accordingly, the integrand of I_0 about y_l , we obtain

$$I_0 \approx \int_{y_l-r}^{y_l+r} \frac{c_i Q'_l}{\left[U_l + \frac{1}{2} U''_l (y-y_l)^2 - c^{(\text{loc})} \right]^2} |\phi^{(\text{loc})}|^2 dy,$$

where the superscript (loc) marks the solution of the “local” eigenvalue problem (11), (7). Observe that, by definition, the localization radius r is sufficiently large to include all of the region, where the integrand is of order one. Hence, if we shift the limits of integration to $\pm\infty$, I_0 will not change significantly,

$$I_0 \approx c_i Q'_l \int_{-\infty}^{\infty} \frac{|\phi^{(\text{loc})}|^2}{\left[U_l + \frac{1}{2} U''_l (y-y_l)^2 - c^{(\text{loc})} \right]^2} dy. \quad (16)$$

Next, observe that the integrands of I_- and I_+ are small everywhere except for the *critical layers* (regions surrounding critical levels), where the denominator $U-c$ is small due to the smallness of c_i . Moreover, since the contribution of a critical level is proportional to the amplitude of the eigenfunction there, and since the eigenfunction decays exponentially with distance from the localization point, we need to take into account only the critical layer that is the closest to y_l . Assuming it to be located to the left of y_l , we shall estimate I_- . Since the width of the critical layer is small (comparable to $|c_i|$), we can put there

$$U(y) \approx U_c + U'_c (y-y_c), \quad Q'(y) \approx Q'_c, \quad \phi(y) \approx \phi_c,$$

where the subscript c marks the value of the corresponding function at $y=y_c$. Then, I_- becomes

$$I_- \approx Q'_c |\phi_c|^2 \int_{-\infty}^{y_l-r} \frac{c_i}{[U_c + U'_c (y-y_c) - c_r]^2 + c_i^2} dy.$$

The integral can now be evaluated. Replacing its upper limit with ∞ (recall that y_c and y_l are separated by a distance greater than r), we obtain

$$I_- \approx \pi(\text{sign } c_i) \frac{Q'_c |\phi_c|^2}{|U'_c|}. \quad (17)$$

Finally, note that I_+ may include only those critical levels that are farther from y_l than those included in I_- . Thus, the former is negligible in comparison with the latter, and we can simply put

$$I_+ \approx 0. \quad (18)$$

Now, substitution of (16)–(18) into (15) yields

$$c_i Q'_l \int_{-\infty}^{\infty} \frac{|\phi^{(\text{loc})}|^2}{\left[U_l + \frac{1}{2} U''_l (y-y_l)^2 - c^{(\text{loc})} \right]^2} dy + \pi(\text{sign } c_i) \frac{Q'_c |\phi_c|^2}{|U'_c|} \approx 0. \quad (19)$$

(19) should be treated as an equation for c_i . One can see that, if

$$\frac{Q'_c}{Q'_l} < 0, \quad (20)$$

(19) has two solutions, corresponding to two complex-conjugate eigenvalues, one of which is unstable. On the other hand, if (20) does *not* hold, no solutions exist for c_i . In this case, our eigenvalue problem does not have eigenvalues at all (stable or unstable), which should be interpreted as stability.

Note that the instability criterion (20) implies the knowledge of the coordinate y_c of the critical level. In order to find it, recall that, to the leading order, $\text{Re } c \approx c^{(\text{loc})}$ and $c^{(\text{loc})} \approx U_l$, which results in the following approximate equation for y_c ,

$$U(y_c) = U_l \quad (21)$$

[if (21) has several solutions, we should choose the one nearest to y_l].

Finally, we note that the above calculation is valid only if

$$U'_c \neq 0$$

[otherwise I_- would be infinite—see (17)].

V. DISCUSSION

In this section, we shall briefly discuss the instability condition (20).

(1) Observe that (20) guarantees instability only provided the existence of a solution to the “local” eigenvalue problem (11), (7). Thus, the full instability criterion consists of the localization condition (12), instability condition (20), and Eq. (21) for the position of the critical level.

(2) It should be emphasized that this criterion applies only to *short-wave* disturbances, and should be viewed, therefore, as *sufficient*, but not *necessary*, condition of instability. In other words, a flow may “fail” it, yet be unstable with respect to *long-wave* disturbances.

(3) Observe that, in a sense, our (sufficient) instability criterion complements the standard (necessary) criterion based on monotonicity of PV (see Ref. 1). Indeed, a change of sign of the PV gradient does not necessarily make the flow unstable. In order to guarantee instability, the PV gradient should have opposite signs at “important” points, such as the localization point and critical level.

(4) Note that (19) includes an unknown quantity, ϕ_c , and therefore does not allow one to actually find c_i . Indeed, ϕ_c cannot be determined using the “local” approximation (which does not work near the critical level). It can be calculated only through the method of matched asymptotic ex-

pansions, which is associated, in the present case, with cumbersome calculations. We shall not do this, as our main target [the instability criterion (12), (20)–(21)] could have been derived without it.

(5) Observe that, even though (19) cannot provide a solution for c_i , it can be used to show that c_i is exponentially small. Indeed, it follows from (19) that

$$c_i \sim |\phi_c|^2.$$

Next, recall that everywhere, except for the localization point and critical level, the exact equation (5) can be reduced to

$$\frac{d^2 \phi}{dy^2} - k^2 \phi \approx 0.$$

The solution of this equation is a linear combination of e^{kz} and e^{-kz} . Then, recalling that ϕ decays with distance from y_l , we conclude that

$$\phi(y) \sim e^{-k|y-y_l|}, \quad \text{if } |y-y_l| > r,$$

where r is the localization radius. This argument leads to the estimate

$$c_i \sim e^{-2k|y_c-y_l|},$$

as required.

(6) Observe that, if $U(y)$ has no extrema (is monotonic), disturbances do not have a point to be localized about. As a result, no modes, stable or unstable, exist in the problem, which should be interpreted as stability with respect to short-wave perturbations.

Interestingly, this conclusion coincides with the corresponding result obtained in Ref. 4 for the (similar) case of flows in a nonrotating fluid with inverse stratification (where the density grows upward).

(7) If $U(y)$ is even, our derivation of (20), strictly speaking, fails. In this case, *two* (not one) critical levels are to be taken into account, as they are located at exactly the same distance from the localization point. However, their contributions are identical, hence, the criterion remains the same.

VI. COMPARISON WITH THE EXACT SOLUTION

In order to verify our asymptotic analysis, the exact eigenvalue problem (5), (7) has been solved numerically, using a method described in Appendix D. Two particular cases have been considered: a single jet and two coupled jets. In both cases, for the sake of simplicity, the beta effect was neglected,

$$\alpha = 0, \quad (22)$$

and the slope of topography was represented by a Gaussian function,

$$\frac{dH}{dy} = -S_0 \exp\left(-\frac{y^2}{2L^2}\right) \quad (23)$$

[(23) can be viewed as a model of a continental shelf, of nondimensional width L , and nondimensional depth variation $\sqrt{\pi/2}LS_0$].

A. Single jet

In this case, the flow was represented by the “unit Gaussian jet,”

$$U = \exp\left(-\frac{y^2}{2}\right) \quad (24)$$

(recall that the problem has been nondimensionalized using the amplitude and characteristic width of the jet). Applying the localization condition (12) to the particular case (22)–(24), one can see that localized short-wave modes exist only if

$$S_0 < -1. \quad (25)$$

Since these modes are associated with the absolute maximum of $U(y)$, and since their phase speeds exceed $\max\{U(y)\}$, they have no critical levels. Therefore, they are stable.

This conclusion has been verified numerically. A sufficiently fine “grid” on the (L, S_0) plane was chosen, and if a mode was found for a particular node of the grid, it was marked with a dot [see Fig. 3(a)]. One can see that, with the exception of the small- L region, the asymptotic condition (25) agrees with the exact results remarkably well. Moreover, the asymptotic and exact results agree not only for *large* k , but for *moderate* k as well (as long as k is large enough for all nonlocalized modes to disappear). (Apart from the stable, localized modes with real c , slightly exceeding the maximum velocity of the jet, the problem may include nonlocalized modes. Those exist, however, only for moderate k and are ignored in the present paper.) In practice, Fig. 3(a) looks exactly the same way for any $k \geq 2$.

As for the small- L region, the discrepancy is likely to be blamed on the numerical technique, rather than the asymptotic one. There are strong indications that a solution exists in those cases as well (provided that $S_0 < -1$, of course), but the numerical results are a little inconclusive.

We have also compared the asymptotic and exact dispersion curves, i.e., the dependencies of c on k , for a particular case,

$$L_0 = 1, \quad S_0 = -2$$

[see Fig. 3(b)]. Again, the asymptotic and numerical solutions agree remarkably well even if $k = \mathcal{O}(1)$. For example, the relative error of the asymptotic solution for $k = 1$ is only 8%.

B. Two coupled jets

In this case, the flow was represented by

$$U = \exp\left[-\frac{(y-d)^2}{2}\right] + \exp\left[-\frac{(y+d)^2}{2}\right], \quad (26)$$

which describes two Gaussian jets of equal amplitudes and widths, separated by a distance of $2d$. It turns out that the stability properties of (26) strongly depend on d .

If $|d| \leq 1$, (26) has a single extremum and thus describes a single jet (this case is similar to the one examined above and will not be considered further).

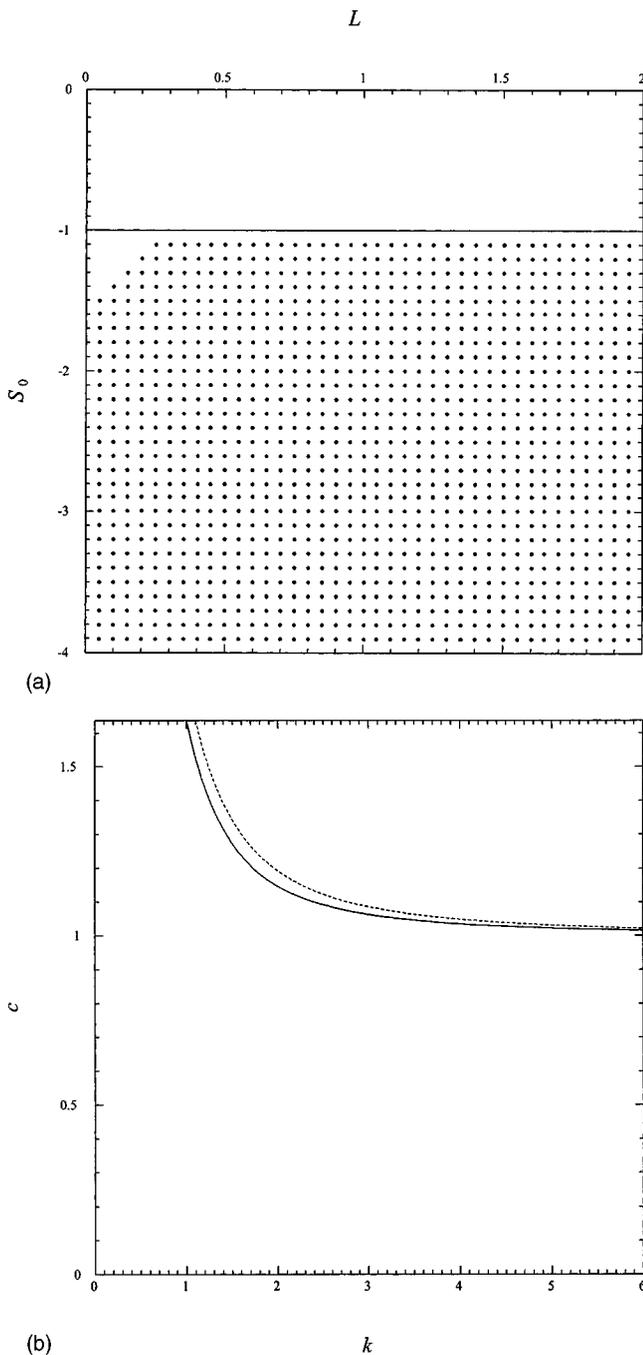


FIG. 3. Disturbances in a single jet (22)–(24). (a) The existence of localized modes with $k=2$, for various values of the topography parameters L and S_0 . Dots show values of (L, S_0) for which a mode has been found using the exact eigenvalue problem (5), (7). The solid horizontal line shows the asymptotic boundary of the existence region [determined by (25)]. (b) The dispersion curve of a stable mode localized near the “tip” of jet. The solid line shows the exact solution [computed using the eigenvalue problem (5), (7)]; the dotted line shows the asymptotic solution [computed using the “local” problem (11), (7)].

If $|d| > 1$, (26) has two maxima and a minimum in between. Observe that modes localized near the maxima cannot be unstable, as they do not have critical levels (the phase speed of a mode localized near a maximum of U exceeds slightly the local maximum velocity, which automatically rules out critical levels elsewhere). Thus, unstable modes can be localized only near the minimum of U (see Fig. 2), where

$$U''_l > 0. \tag{27}$$

Then, the localization condition (12) yields

$$Q'_l > 0, \tag{28}$$

and the instability condition (20) yields

$$Q'_c < 0. \tag{29}$$

Note that (27)–(29) apply to *any* $U(y)$ with a minimum [not only to profile (26) considered here].

Next, recalling definition (8) of the PV gradient Q' and adopting it for the case $\alpha=0$, we rewrite (27)–(29) as

$$0 < U''_l < S_l, \quad U''_c > S_c, \tag{30}$$

where

$$S = - \frac{dH}{dy}$$

characterizes the slope of topography. Finally, we recall that the velocities at the localization point and critical level are, to the leading order, equal to

$$U_l = U_c \tag{31}$$

[which follows from the fact that the phase speed of the mode is close to U_l —see (21)].

Observe that, if $S(y)$ is sign definite [which is the case for “our” S —see (23)], the instability criterion (30)–(31) implies that

$$U''_c > 0. \tag{32}$$

This condition imposes a restriction on the velocity profile. Solving (numerically) (31)–(32) for the coupled-jet profiles (26), we obtain

$$d > 1.542. \tag{33}$$

Thus, the profiles that do *not* satisfy this condition are stable (with respect to short-wave disturbances) *regardless of topography*.

In order to make the situation “richer,” we choose d from range (33), namely,

$$d = 1.6 \tag{34}$$

(d should not be too large, as this would make the growth rate difficult to compute due to its exponential decay with distance between the localization point and critical level).

The asymptotic instability criterion (30)–(31) was compared to the exact solution for the particular case (22)–(23), (26), (34). As before, a sufficiently fine “grid” on the (L, S_0) plane was chosen, and if an unstable mode was found for a particular node of the grid, it was marked with a dot (see Fig. 4). One can see that the asymptotic and exact results agree fairly well, and even more so, since Fig. 4 was computed for $k=2$. It should be admitted though that a moderate k was mainly chosen because of numerical difficulties associated with large k , due to the exponential smallness of the growth rate and the fact that the phase speed is close to the local value of U (the difficulties are particularly severe for small L).

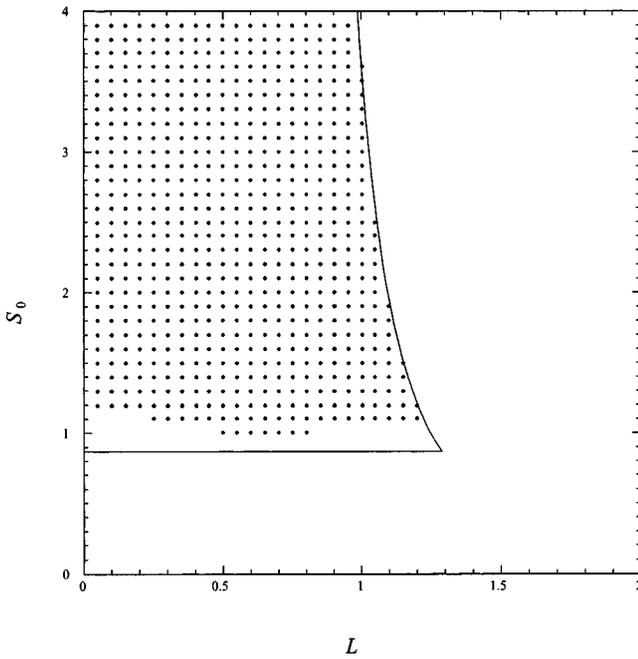


FIG. 4. Instability of two coupled jets, described by (22)–(23), (26), (34), for various values of the topography parameters L and S_0 , and disturbances with $k=2$. Dots show values of (L, S_0) for which an unstable mode has been found using the exact eigenvalue problem (5), (7). Solid lines show the asymptotic curves of marginal stability [determined by the asymptotic conditions (30)–(31)].

Thus, the numerical and asymptotic methods complement each other in this problem (the former works for moderate k and the latter works for large k).

VII. SUMMARY AND CONCLUDING REMARKS

Summarizing the results of this paper, we can formulate the following criterion of (short-wave) instability.

The flow characterized by a velocity profile $U(y)$ and PV profile $Q(y)$ is unstable if the following is true:

- (1) $U(y)$ has a local extremum, say at y_l [$U'(y_l)=0$, $U''(y_l)\neq 0$], where the PV gradient Q' and the vorticity gradient $-U''$ are of opposite signs;
- (2) there exist one or more points y_c (critical levels), such that $U(y_c)=U(y_l)$;
- (3) the closest to y_l critical level is not an extremum ($U'(y_c)\neq 0$), and the PV gradient there, $Q'(y_c)$, is opposite in sign to $Q'(y_l)$.

Condition (1) guarantees localization (see Sec. III), whereas conditions (2)–(3) guarantee that a critical level exists and makes the localized short-wave mode unstable (see Sec. IV). It has also been proven that, if y_l is a maximum of $U(y)$, the phase speed c of the mode is greater than U_l , and if y_l is a minimum, the phase speed is less than U_l . This property guarantees that the mode does not have critical levels near the localization point. Note that, in either case, the difference between c and U_l is small, which is why y_c , defined in condition (2), is interpreted as a critical level.

We shall also summarize briefly what happens if a flow does not satisfy some of the conditions above.

- (i) If a flow has a single extremum, the modes, even if exist [i.e., if condition (1) holds], do *not* have critical levels. In this case, the flow is stable with respect to short-wave disturbances.

It should be noted though that flows with one extremum can still be unstable with respect to long-wave disturbances, which is why our (short-wave) criterion is a sufficient, but not necessary, condition of instability.

- (ii) Monotonic flows do not have points where modes could be localized about. In this case, no short-wave modes (stable or unstable) exist in the problem, which should be regarded as short-wave stability.

Observe that our results appear to be applicable to jets regardless of the physical setting. In particular, we can readily generalize them for equivalently barotropic flows, i.e., flows in a thin layer of fluid on top of a thick layer. We only need to modify the expression for the PV gradient: instead of (6), it should be

$$Q' = \beta + \frac{1}{R_d^2} U - \frac{d^2 U}{dy^2},$$

where R_d is the so-called internal deformation radius.

Our approach can also be applied to the classical problem of instability of *nonrotating* flows (such as the Poiseuille or Couette flows). In this case, the vorticity and potential vorticity coincide, hence, the localization condition [condition (1)] never holds. Thus, for this setting, instability always has a short-wave cutoff.

Not dwelling on examples, for which generalization of the present results is not obvious (such as baroclinic jets on the beta plane), we note that there is an important case where our approach is clearly applicable and has a potential of producing important results. Namely, the stability of jets in magnetized isothermal plasma is described by an equation similar to our equation (5),

$$\frac{d^2 \phi}{dy^2} - \left[k^2 + \frac{U_{yy}}{U-c} - \frac{N^2}{(U-c)(U+V-c)} \right] \phi = 0, \quad (35)$$

where $U(y)$ is the velocity of the flow, $V(y)$ is the so-called gradient-B drift (due to the magnetic field), and $N(y)$ is an equivalent of the Väsälä frequency (it depends on both density gradient and magnetic field). [Equation (35) was brought to my attention by Jens Juul Rasmussen (see Ref. 6).] In this case, short-wave modes can be localized near the extrema of $U(y)$ or $U(y)+V(y)$, whereas critical levels may occur at points where $U(y_c)=\text{Re } c$ or $U(y_c)+V(y_c)=\text{Re } c$. These possibilities make the problem extremely rich and interesting.

APPENDIX A: NONEXISTENCE OF SOLUTIONS TO THE EIGENVALUE PROBLEM (10), (7)

In terms of

$$\xi = k(y - y_l), \quad C = \frac{k(c - U_l)}{U_l'},$$

(10), (7) become

$$\phi_{\xi\xi} - \phi + \frac{A}{\xi - C} \phi = 0, \quad (A1)$$

$$\phi \rightarrow 0, \quad \text{as } \xi \rightarrow \pm \infty, \quad (\text{A2})$$

where

$$A = \frac{kQ'_l}{U'_l}$$

and $\phi_\xi = d\phi/d\xi$.

Theorem A1: *Problem (A1), (A2) does not have complex eigenvalues.*

This theorem is, essentially, a particular case of Rayleigh's theorem for flows with monotonic PV, and it can be proven in a similar manner [by multiplying (A1) by ϕ^* and integrating over $-\infty < \xi < \infty$].

Observe that, by virtue of Theorem A1, Eq. (A1) has a singularity at $\xi = C$ (a nonzero C_i would prevent $\xi - C$ from

vanishing for all real ξ). In order to "resolve" the singularity, we introduce infinitely small damping, i.e., add infinitesimal imaginary correction to C ,

$$\phi_{\xi\xi} - \phi + \frac{A}{\xi - C - i0} \phi = 0. \quad (\text{A3})$$

This equation will now replace (A1).

In order to clarify the behavior of ϕ at $\xi = C$, observe that

$$\phi = \text{const}_1 \phi_1 + \text{const}_2 \phi_2, \quad (\text{A4})$$

where the linearly independent solutions $\phi_{1,2}$ can be chosen such that

$$\begin{aligned} \phi_1 &= (\xi - C - i0) + O[(\xi - C - i0)^2], \\ \phi_2 &= 1 - A(\xi - C - i0)\ln(\xi - C - i0) + O[\ln(\xi - C - i0)(\xi - C - i0)^2] \end{aligned} \quad \text{as } \xi \rightarrow C$$

(these expansions were obtained using the Frobenius method). Note that ϕ_1 is regular and, therefore, $i0$ can be omitted, whereas ϕ_2 can be rearranged using the formula $\ln \xi = \ln|\xi| + \arg \xi$,

$$\begin{aligned} \phi_1 &= (\xi - C) + O[(\xi - C)^2], \\ \phi_2 &= 1 - A(\xi - C)(\ln|\xi - C| + i\pi) + O[(\xi - C)^2 \ln|\xi - C|], \end{aligned} \quad \text{as } \xi \rightarrow C - 0, \quad (\text{A5})$$

$$\begin{aligned} \phi_1 &= (\xi - C) + O[(\xi - C)^2], \\ \phi_2 &= 1 - A(\xi - C)(\ln|\xi - C| + 2i\pi) + O[(\xi - C)^2 \ln|\xi - C|] \end{aligned} \quad \text{as } \xi \rightarrow C + 0. \quad (\text{A6})$$

Now, we can prove the following theorem.

Theorem A2: *The eigenfunctions of problem (A3), (A2), if exist, vanish at $\xi = C$.*

Since the coefficients of (A3) are regular and real for $\xi \neq C$, the Wronskian,

$$W = \phi_\xi \phi^* - \phi_\xi^* \phi,$$

is constant within the intervals $(-\infty, C)$ and (C, ∞) , but may have a jump at $\xi = C$. Observe also that the boundary condition implies $\lim_{\xi \rightarrow \pm \infty} W = 0$, hence,

$$W(C+0) = 0, \quad W(C-0) = 0. \quad (\text{A7})$$

At the same time, the jump of W at $\xi = C$ can be calculated using (A4)–(A6),

$$W(C+0) - W(C-0) = -2i\pi A |\phi(C)|^2. \quad (\text{A8})$$

A comparison of (A7) and (A8) yields $\phi(C) = 0$, as required.

It follows from Theorem A2 that const_2 in equality (A4) is zero, hence, $\phi = \text{const}_1 \phi_1$, i.e.,

$$\phi = O(\xi - C), \quad \text{as } \xi \rightarrow C. \quad (\text{A9})$$

This condition will be used for proving the next theorem.

Theorem A3: *The eigenvalue problem (A3), (A2) has no solutions.*

First, we shall prove this theorem for $A > 0$. Multiply (38) by ϕ^* and integrate it over $-\infty < \xi < C$. Integration by parts yields

$$\begin{aligned} &(\phi^* \phi_\xi)_{\xi=C} - (\phi^* \phi_\xi)_{\xi \rightarrow -\infty} \\ &- \int_{-\infty}^C \left(|\phi_\xi|^2 + |\phi|^2 - \frac{A}{\xi - C} |\phi|^2 \right) d\xi = 0, \end{aligned} \quad (\text{A10})$$

where the integral converges due to (A9) (accordingly, the regularizing term, $i0$, has been omitted). The first term in (A10) vanishes due to Theorem A2 and the second term vanishes due to the boundary condition (A2). We obtain

$$\int_{-\infty}^C \left(|\phi_\xi|^2 + |\phi|^2 - \frac{A}{\xi - C} |\phi|^2 \right) d\xi = 0. \quad (\text{A11})$$

Since the integrand of (A11) is positive everywhere in the region of integration, the integral may not equal zero, which contradiction can be resolved only if ϕ does not exist, as required.

The case $A < 0$ should be treated in the same manner, but with different limits of integration in the analog of (A10) [(C, ∞) instead of $(-\infty, C)$].

APPENDIX B: PROPERTIES OF THE EIGENVALUE PROBLEM (11), (7)

In terms of

$$\xi = k(y - y_l),$$

and

$$C = \frac{2k^2(c - U_l)}{U_l''}, \tag{B1}$$

(11), (7) can be rewritten as

$$\phi_{\xi\xi} - \phi + \frac{B}{\xi^2 - C} \phi = 0, \tag{B2}$$

$$\phi \rightarrow 0, \text{ as } \xi \rightarrow \pm\infty, \tag{B3}$$

where

$$B = \frac{2Q_l'}{U_l''}. \tag{B4}$$

Physically, B is proportional to the ratio of the vorticity gradient $-U''$ and PV gradient Q' .

Theorem B1: *Problem (B2)–(B3) has no complex eigenvalues.*

Similar to Theorem A1, this is a particular case of Rayleigh’s theorem for flows with monotonic PV; but we shall prove it anyway, as one of the intermediate results will be used later.

Multiply (B2) by ϕ^* and integrate over $-\infty < \xi < \infty$. Integrating by parts and taking into account the boundary condition (B3), we obtain

$$\int_{-\infty}^{\infty} \left(|\phi_{\xi}|^2 + |\phi|^2 - \frac{B}{\xi^2 - C} |\phi|^2 \right) d\xi = 0. \tag{B5}$$

Taking the imaginary part of (B5), one can see that it holds only if $C_i = 0$, as required.

Theorem B2: *If an eigenvalue of problem (B2)–(B3) is positive, the corresponding eigenfunction $\phi(\xi)$ vanishes at $\xi = \pm\sqrt{C}$.*

The proof of this theorem is similar to that of Theorem A2.

Theorem B3: *Problem (B2)–(B3) has no positive eigenvalues.*

The proof of this theorem is similar to that of Theorem A3, but the integration in the analog of (A10) should be over $(-\sqrt{C}, \sqrt{C})$ for $B > 0$, and (\sqrt{C}, ∞) for $B < 0$.

Further properties of the eigenvalue problem (B2)–(B3) depend on the sign of B . In what follows, we shall demonstrate that no eigenvalues exists for $B < 0$, and at least one (negative) eigenvalue exists for $B > 0$.

1. Nonexistence of solutions for $B < 0$

Theorem B4: *The eigenvalue problem (B2)–(B3) with $B < 0$ has no solutions.*

For $C > 0$, this theorem has been proven as a part of Theorem B3; and for $C < 0$, it follows from (B5) [if $C < 0$ and $B < 0$, the integrand in (B5) would be positive everywhere, and the integral could not be equal to zero].

Thus, by virtue of Theorem B4, (B2)–(B3) may have solutions only if

$$B > 0, \tag{B6}$$

and, by virtue of Theorem B3, these eigenvalues may not be positive,

$$C < 0. \tag{B7}$$

Substituting (B4) into (B6) and (B1) into (B7), we obtain (12), (13), as required.

It remains to be shown that condition (B6) guarantees the existence of at least one solution to the eigenvalue problem (B2)–(B3).

2. Existence of a solution for $B > 0$

Theorem B5: *With B increasing from 0 to ∞ , the number of eigenvalues of (B2)–(B3) may not decrease.*

Observe that, with changing B , the eigenvalues of (B2)–(B3) may not become complex (due to Theorem B1) or positive (due to Theorem B3). Hence, Theorem B5 can only be violated if eigenvalues “escape” through the points $C = 0$ or $C = -\infty$.

In order to prove that neither is possible, it is sufficient to demonstrate that

$$\frac{dC}{dB} < 0, \text{ if } C \rightarrow -0, \tag{B8}$$

and

$$\left| \frac{dC}{dB} \right| < \infty, \text{ if } B < \infty. \tag{B9}$$

Condition (B8) guarantees that the eigenvalues are “repelled” if they approach $C = 0$, and (B9) guarantees that eigenvalues may not escape to infinity for finite B .

To prove (B8) and (B9), differentiate the eigenvalue problem (B2)–(B3) with respect to B ,

$$\begin{aligned} \left(\frac{\partial \phi}{\partial B} \right)_{\xi\xi} - \frac{\partial \phi}{\partial B} + \frac{B}{\xi^2 - C} \frac{\partial \phi}{\partial B} + \frac{1}{\xi^2 - C} \phi \\ + \frac{B}{(\xi^2 - C)^2} \frac{dC}{dB} \phi = 0, \end{aligned} \tag{B10}$$

$$\frac{\partial \phi}{\partial B} \rightarrow 0, \text{ as } \xi \rightarrow \pm\infty.$$

Then, multiply (B10) by ϕ^* and integrate over $-\infty < \xi < \infty$. Integrating by parts (twice) and using the boundary conditions for ϕ and $\partial\phi/\partial B$, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\phi_{\xi\xi} - \phi + \frac{B}{\xi^2 - C} \phi \right) \frac{\partial \phi}{\partial B} d\xi + \int_{-\infty}^{\infty} \frac{|\phi|^2}{\xi^2 - C} d\xi \\ + \frac{dC}{dB} \int_{-\infty}^{\infty} \frac{B}{(\xi^2 - C)^2} |\phi|^2 d\xi = 0. \end{aligned}$$

Observe that the expression in brackets in the first term is identically zero [due to (B2)], whereas the second term can be expressed using identity (B5). After simple algebra, we obtain

$$\frac{dC}{dB} = - \frac{\int_{-\infty}^{\infty} (|\phi_{\xi}|^2 + |\phi|^2) d\xi}{B^2 \int_{-\infty}^{\infty} \frac{|\phi|^2}{(\xi^2 - C)^2} d\xi}. \tag{B11}$$

First, $dC/dB < 0$ (for all C), which guarantees the validity of (B8). Second, the nonzero denominator and convergence of all integrals in (B11) (for $C < 0$) guarantee the validity of (B9).

Thus (real, negative), eigenvalues cannot escape through $C = 0$ or $C = -\infty$, and their number may not decrease, as required.

Theorem B6: *The eigenvalue problem (B2)–(B3) with $B > 0$, has at least one solution.*

Observe that, as shown in Appendix C, (B2)–(B3) have a solution, such that

$$C \rightarrow -\frac{\pi^2 B^2}{4}, \quad \text{as } B \rightarrow +0$$

[see (C6) and, also, Fig. 5(a)]. Then, due to Theorem B5, this eigenvalue may not disappear for all $B > 0$, as required.

Theorem B6 has been verified and complemented by numerical calculations. It was confirmed that the number of eigenvalues does not decrease with increasing B [moreover, it actually grows—see Figs. 5(b), 5(c)].

APPENDIX C: ASYMPTOTIC ANALYSIS OF THE EIGENVALUE PROBLEM (11), (7)

In this appendix, we shall use the same form of (11), (7) as in Appendix B, i.e., Eqs. (B2)–(B3). We shall consider two limits: $B \rightarrow +0$ and $B \rightarrow +\infty$.

1. The limit $B \rightarrow +0$

In this case, it is convenient to rescale the problem as follows:

$$\eta = \frac{\xi}{B}, \quad \hat{C} = \frac{C}{B^2}. \tag{C1}$$

The substitution of (C1) into (B2)–(B3) yields

$$\phi_{\eta\eta} - B^2 \phi + \frac{B}{\eta^2 - \hat{C}} \phi = 0, \tag{C2}$$

$$\phi \rightarrow 0, \quad \text{as } \eta \rightarrow \pm\infty. \tag{C3}$$

Note that our eigenvalue problem is invariant with respect to the replacement of $\eta \rightarrow -\eta$; hence, $\phi(\eta)$ is either even or odd. The former case will be considered first.

It can be readily seen from (C2)–(C3), that an even ϕ can be assumed to satisfy

$$\phi \rightarrow e^{\mp B\eta}, \quad \text{as } \eta \rightarrow \pm\infty. \tag{C4}$$

(C4) will be used as a boundary condition instead of (C3).

Next, we seek a solution of the form

$$\phi = \phi^{(0)} + B\phi^{(1)} + \dots$$

Substituting this expansion into (C2), (C4), we obtain, in the first two orders,

$$\begin{aligned} \phi_{\eta\eta}^{(0)} &= 0, \quad \phi^{(0)} \rightarrow 1, \quad \text{as } \eta \rightarrow \pm\infty, \\ \phi_{\eta\eta}^{(1)} + \frac{1}{\eta^2 - \hat{C}} \phi^{(0)} &= 0, \quad \phi^{(1)} \rightarrow \pm\eta, \quad \text{as } \eta \rightarrow \pm\infty. \end{aligned} \tag{C5}$$

Clearly, $\phi^{(0)} = 1$. Substituting $\phi^{(0)}$ into (C5), we see that it has a solution only if \hat{C} satisfies

$$\int_{-\infty}^{\infty} \frac{d\eta}{\eta^2 - \hat{C}} = 2.$$

This equation has only one solution:

$$\hat{C} = -\frac{\pi^2}{4}.$$

Changing back from \hat{C} to C , we obtain

$$C = -\frac{\pi^2 B^2}{4}. \tag{C6}$$

This (asymptotic) expression has been tested against the exact solution of (B2)–(B3) computed numerically [see Fig. 5(a)]. One can see that the asymptotic and exact results agree well for sufficiently small B ($B \lesssim 0.01$).

Finally, a similar calculation shows that the limit $B \rightarrow +0$ does not admit modes with odd ϕ .

2. The limit $B \rightarrow +\infty$

In this case, it is convenient to rescale the problem as follows:

$$\eta = \frac{\xi}{B^{1/4}}, \quad \hat{C} = \frac{C+B}{B^{1/2}}. \tag{C7}$$

Substituting (C7) into (B2)–(B3), we obtain

$$\phi_{\eta\eta} - \frac{\eta^2 - \hat{C}}{1 + B^{-1/2}(\eta^2 - \hat{C})} \phi = 0, \tag{C8}$$

$$\phi \rightarrow 0, \quad \text{as } \eta \rightarrow \pm\infty. \tag{C9}$$

Expanding (C8) in $B^{-1/2}$ and keeping the leading order only, we obtain

$$\phi_{\eta\eta} + (\hat{C} - \eta^2) \phi \approx 0. \tag{C10}$$

This is the equation of parabolic cylinder, and the eigenvalues of problem (C10), (C9) can be readily found,

$$\hat{C} = 2m - 1, \quad m = 1, 2, 3, \dots$$

The infinite number of the asymptotic eigenvalues indicates that the number of the exact eigenvalues grows as $B \rightarrow \infty$ [this conclusion is corroborated by the numerical solution of the eigenvalue problem (B2)–(B3)—see Figs. 5(b), 5(c)].

Changing back from \hat{C} to C , we have

$$C = -B + (2m - 1)\sqrt{B}. \tag{C11}$$

This (asymptotic) expression has been tested against the exact solution of (B2)–(B3) computed numerically [see Fig. 5(c)]. It turned out that the accuracy of the asymptotic solution deteriorates dramatically for higher mode numbers. For example, in order to get the asymptotic and exact results to agree for $m = 3$, one needs to consider $B \gtrsim 150$.

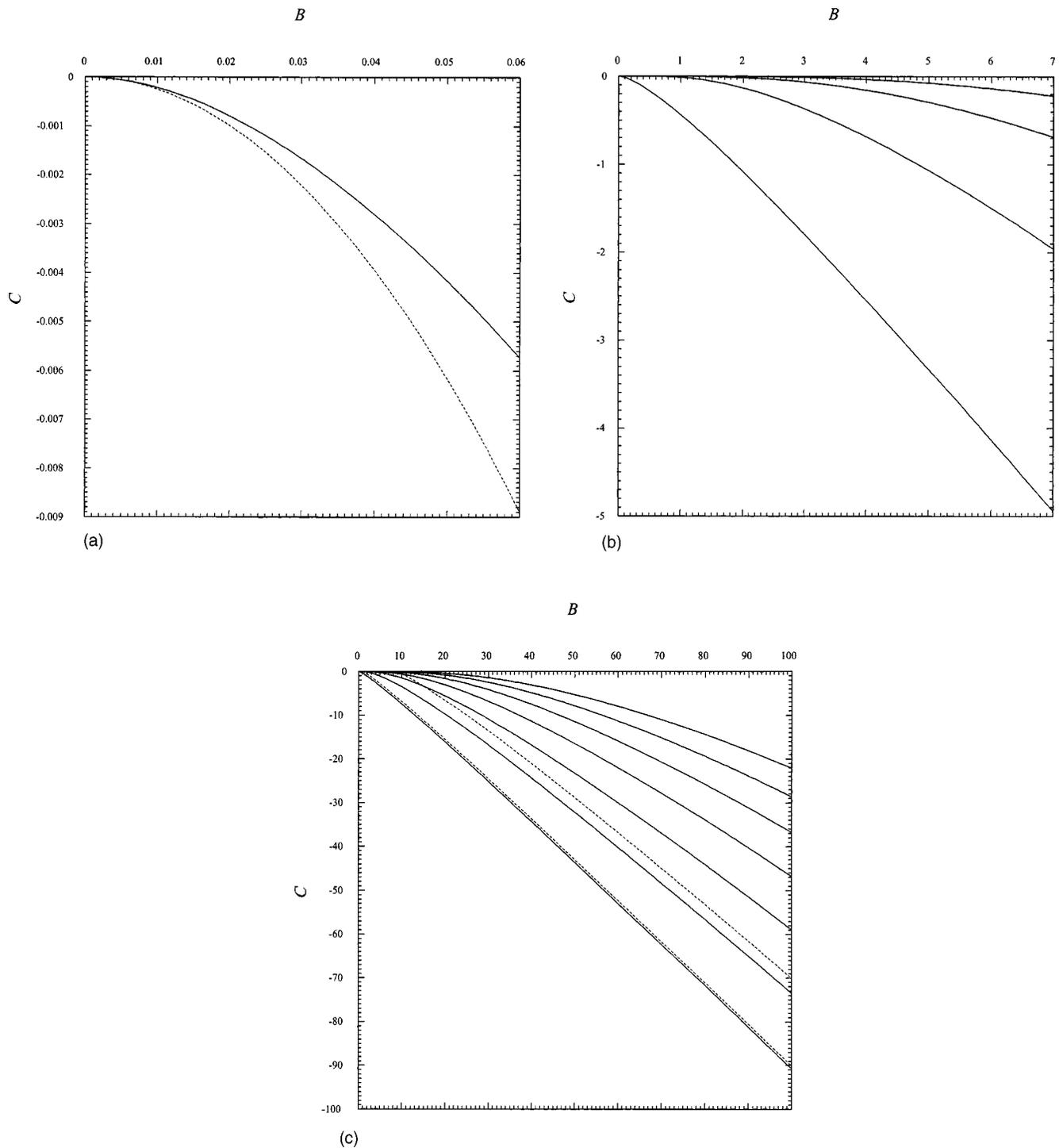


FIG. 5. The solution of the eigenvalue problem (B2)–(B3). The exact eigenvalue versus parameter B is shown in a solid line. The curves are marked with the corresponding mode number. (a) $B \rightarrow +0$. The dotted line shows the asymptotic solution (C6). (b) An illustration of how “new” modes appear with increasing B . (c) $B \rightarrow +\infty$. The dotted line shows the asymptotic solution (C11) for the first two modes (the higher modes agree with the asymptotic results only for larger B). It can be seen that the number of modes grows with increasing B .

APPENDIX D: THE NUMERICAL METHOD

In order to compute the eigenvalue c of the problem (5), (7), two particular solutions, ϕ_- and ϕ_+ , were “shot” using the Runge–Kutta method from $y = -\infty$ and $y = \infty$, using the asymptotics

$$\phi_{\pm} \rightarrow e^{\mp ky}, \quad \text{as } y \rightarrow \pm \infty$$

[these boundary conditions follow from (5) as $y \rightarrow \pm \infty$]. The Wronskian of these solutions at $y=0$,

$$W(c) = \left(\frac{d\phi_+}{dy} \phi_- - \frac{d\phi_-}{dy} \phi_+ \right)_{y=0},$$

was fed to a root-finding routine based on the secant method, yielding a solution for c .

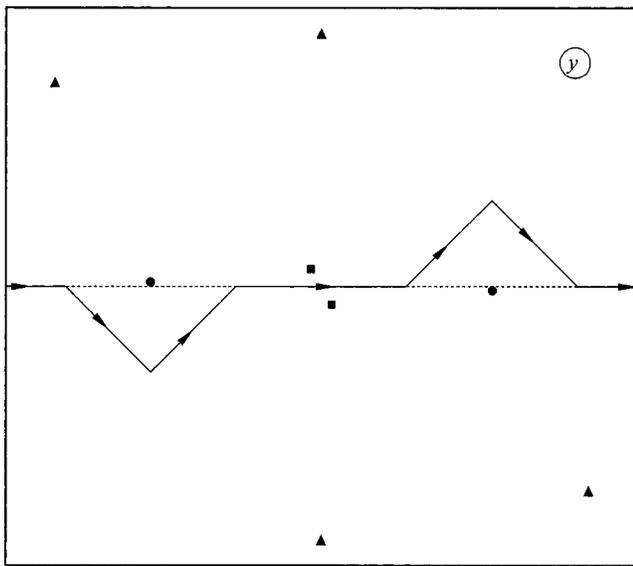


FIG. 6. A typical path of numerical integration of Eq. (5) on the plane of complex y . Circles show the “physical” critical levels and squares show the “nonphysical” critical levels associated with the localization point. All other “nonphysical” critical levels are shown by triangles.

If, however, c_i is small and

$$\min\{U(y)\} < c_r < \max\{U(y)\}, \quad (\text{D1})$$

Eq. (5) becomes almost singular at critical levels, where

$$U(y_c) = c_r. \quad (\text{D2})$$

In such cases, one can extend (5) and its solution to the plane of complex y and modify the path of integration in such a way that it bypasses the critical level (see Refs. 7 and 8). One would still have to keep the endpoints fixed, and also make sure that the modified path can be transformed back to the real axis without touching any of the singular points of the equation. This would guarantee that the solution would arrive at its final destination with the “correct” value.

In order to illustrate this algorithm, consider the case of two coupled Gaussian jets (26), (34),

$$U = \exp\left(-\frac{(y-1.6)^2}{2}\right) + \exp\left(-\frac{(y+1.6)^2}{2}\right), \quad (\text{D3})$$

and $c = 0.55 + 0.03i$ (the real part of c was chosen to be slightly less than the local minimum of U , as is always the case for short-wave modes localized near a minimum). Equation (D3) with the left-hand side determined by (D3) has an infinite number of solutions, for example,

$$y_{1\pm} \approx \pm(2.693 - 0.050i), \quad y_{2\pm} \approx \pm(0.166 + 0.207i),$$

$$y_{3\pm} \approx \pm(4.218 - 2.379i), \quad y_{4\pm} \approx \pm(0.000 - 2.953i).$$

$y_{1\pm}$ correspond to the “physical” critical levels [solutions of Eq. (D1)], whereas the only important “nonphysical” critical levels are $y_{2\pm}$. Those are associated with the localization point [where (D1) is almost satisfied], and they are located close to the real axis. The latter causes severe numerical difficulties, moreover, since the path of integration is “squeezed” between y_{2+} and y_{2-} , it cannot be moved away. The only remedy is to use in this region a smaller step.

All other critical levels are located far away from the real axis and are unimportant. An example of a modified path of integration for this case can be seen in Fig. 6.

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