NOTES AND CORRESPONDENCE

Stability of a Two-Layer Quasigeostrophic Vortex over Axisymmetric Localized Topography

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ABSTRACT

The stability of a quasigeostrophic vortex over a radially symmetric topographic feature (elevation or depression) in a two-layer ocean on the *f* plane is examined. This article's concern is with compensated vortices, that is, those in which the lower layer is at rest (the disturbances, however, are present in both layers). Through numerical solution of the linear normal-mode problem, it is demonstrated that a bottom elevation is a stabilizing influence for a cyclone and a destabilizing influence for an anticyclone, whereas a bottom depression acts in the opposite way. These conclusions are interpreted using an asymptotic theory developed for the case of a thin upper layer. It is demonstrated that an elevation moves the critical level of an unstable mode toward the periphery of the cyclone, which leads to its stabilization. Estimates based on realistic oceanic parameters show that stabilization occurs for relatively small topography (5%–15% of the lower layer's depth).

1. Introduction

The stability of oceanic eddies has been studied for more than 20 years, but the solution of this problem is still unclear. On the one hand, most of the theoretical work on the subject (e.g., Ikeda 1981; Flierl 1988; Helfrich and Send 1988; Carton and McWilliams 1989; Ripa 1992; Killworth et al. 1997; Benilov et al. 1998; Baey and Carton 2002; Benilov 2003; Katsman et al. 2003) indicates that eddies are unstable, whereas observations (e.g., Lai and Richardson 1977) suggest that eddies exist for years. A promising attempt to resolve the contradiction was made by Dewar and Killworth (1995), who considered a Gaussian vortex in the upper layer and a relatively weak Gaussian circulation in the lower layer. It turned out that the "deep flow" can stabilize the eddy or at least considerably weaken its instability. This idea has been developed further by Benilov (2004), who demonstrated that the deep flow, corresponding to constant potential vorticity in the lower layer, stabilizes *all* vortices (not only the Gaussian one); it was also argued that this kind of deep flow arises naturally below oceanic eddies.

It is emphasized, however, that the deep flow is not the only mechanism potentially capable of eddy stabilization. For example, all of the above papers addressed the model of axisymmetric vortex, whereas "real" oceanic eddies are always (at least, slightly) elongated and prone to filamentation—a condition that may contribute to their stability. After all, as shown by Meacham et al. (1994) and McKiver and Dritschel (2003), elliptic vortices can be very resilient, and filaments can wrap around them and prevent leakage, further adding to their survivability (Mariotti et al. 1994).

Another potentially important effect is bottom topography—its influence on eddies has been studied previously by Velasco Fuentes and van Heijst (1994), Mied et al. (1992), Carton and Legras (1994), Swaters (1998), Reznik (1999), and Sutyrin (2001). However, all of these papers address the drift/distortion of the vortex caused by the topographic slope (similar to that caused by the beta effect), and only Nycander and Lacasce (2004) examined how topography affects the vortex's stability: they demonstrated that, in a barotropic ocean, topography can give rise to stable vortices. Thus, given that bottom irregularities occur almost everywhere in the ocean, they can provide an alternative mechanism of stabilization—at least in the regions with no strong currents, where eddies can spend a significant time "hanging" over a particular topographic feature. It remains to be seen, however, whether topography can eliminate *baroclinic* instability, which is usually much stronger than the barotropic instability examined by Nycander and Lacasce (2004).

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This paper explores how an underlying topographic feature can affect baroclinic stability of a vortex. We shall consider the simplest setting possible, based on a quasigeostrophic two-layer model, with both vortex and topography being radially symmetric. In section 2, we shall formulate the problem mathematically, and, in section 3, several examples will be examined numerically. In section 4, an asymptotic theory will be developed, on the basis of which the numerical results will be interpreted and generalized.

2. Formulation

Consider a two-layer ocean with an uneven bottom and rigid lid, so that, at rest, the depth H_{*1} of the upper layer is constant, whereas the depth $H_{*2}(x_*, y_*)$ of the lower layer depends on the spatial polar coordinates r_* and θ_* (asterisks mark dimensional quantities). The flow in the layers will be characterized by the streamfunctions $\psi_{*1,2}(r_*, \theta_*, t_*)$, where t_* is the time variable. The densities of the layers, denoted as $\rho_{1,2}$, are constant and so is the Coriolis parameter f_0 (i.e., we use the f-plane approximation).

To nondimensionalize the problem, we introduce the characteristic upper-layer velocity U_{*1}, the upper-layer deformation radius,

$$L_d = \frac{1}{f_0}\sqrt{\frac{(\rho_2 - \rho_1)gH_{*1}}{\rho_2}},$$

(where g is the acceleration due to gravity), the mean lower-layer depth $\overline{H}_{*2}$, and its characteristic variation Δ_{*2}. We shall assume Δ_{*2} to have a sign, say,

$$\Delta_{*2} > 0 \quad \text{for bottom elevation} \quad \text{and}$$

$$\Delta_{*2} < 0 \quad \text{for bottom depression.}$$

Then, we nondimensionalize the problem,

$$t = \frac{U_{*1}t_*}{L_d}, \quad r = \frac{r_*}{L_d}, \quad \theta = \theta_*, \quad \psi_{1,2} = \frac{\psi_{*1,2}}{L_d U_{*1}}, \quad \text{and}$$

$$D = \frac{\overline{H}_{*2} - H_{*2}}{\Delta_{*2}},$$

where D(x, y) describes the topography. The nondimensional streamfunctions $\psi_{1,2}$ are governed by the standard quasigeostrophic equations on the f plane,

$$\frac{\partial}{\partial t}(\nabla^2\psi_1 - \psi_1 + \psi_2) + J(\psi_1, \nabla^2\psi_1 + \psi_2) = 0 \quad \text{and} \tag{1}$$

$$\frac{\partial}{\partial t}(\nabla^2\psi_2 - \varepsilon\psi_2 + \varepsilon\psi_1) + J(\psi_2, \nabla^2\psi_2 + \varepsilon\psi_1 + \delta D) = 0, \tag{2}$$

where $J(\psi_1, \psi_2)$ is the Jacobian operator, and

$$\varepsilon = \frac{H_{*1}}{\overline{H}_{*2}} \quad \text{and} \quad \delta = \frac{f_0 L_d}{U_{*1}}\frac{\Delta_{*2}}{\overline{H}_{*2}} \tag{3}$$

are the depth ratio and a topography parameter, respectively. We are concerned with linear stability of compensated (i.e., localized in the upper layer), radially symmetric vortices,

$$\psi_1 = \Psi_1(r) + \psi_1'(r, \theta, t) \quad \text{and} \quad \psi_2 = \psi_2'(r, \theta, t), \tag{4}$$

where Ψ_1 describes the vortex, $\psi_{1,2}'$ describe the disturbance, and (r, θ) are polar coordinates. To linearize the governing equations against the background of the vortex solution, substitute (4) into (1) and (2) and omit the nonlinear terms:

$$\frac{\partial}{\partial t}(\nabla^2\psi_1' - \psi_1' + \psi_2') + J(\Psi_1, \nabla^2\psi_1' + \psi_2') + J(\psi_1', \nabla^2\Psi_1) = 0$$

and

$$\frac{\partial}{\partial t}(\nabla^2\psi_2' - \varepsilon\psi_2' + \varepsilon\psi_1') + J(\psi_2', \nabla^2\Psi_1 + \varepsilon\Psi_1) = 0.$$

In this paper, we are concerned with harmonic disturbances (normal modes),

$$\psi_{1,2}'(r, \theta, t) = \mathrm{Re}[\phi_{1,2}(r)e^{ik(\theta - ct)}],$$

where k and c are the azimuthal wavenumber and angular phase speed, respectively. Then, the governing equations yield

$$(cr - V_1)\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi_1}{dr}\right) - \frac{k^2}{r^2}\phi_1 - \phi_1 + \phi_2\right] + \left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rV_1)\right] - V_1\right\}\phi_1 = 0 \quad \text{and} \tag{5}$$

$$cr\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi_2}{dr}\right) - \frac{k^2}{r^2}\phi_2 - \varepsilon\phi_2 + \varepsilon\phi_1\right] + (\varepsilon V_1 + \delta S)\phi_2 = 0, \tag{6}$$

where

$$V_1 = \frac{d\Psi_1}{dr} \quad \text{and} \quad S = \frac{dD}{dr}$$

are the upper-layer swirl velocity and the slope of the bottom, respectively.

Equations (5) and (6) should be supplemented by the usual boundary conditions,

$$\phi_{1,2}(0) = \phi_{1,2}(\infty) = 0. \tag{7}$$

Equations (5)–(7) form an eigenvalue problem, where c is the eigenvalue. If Im$c > 0$, the vortex is unstable.

3. Examples

To find out how topography affects the stability of vortices, consider the Gaussian vortex,

$$V_1(r) = \frac{r}{r_e}\exp\left(-\frac{r^2}{2r_e^2}\right), \tag{8}$$

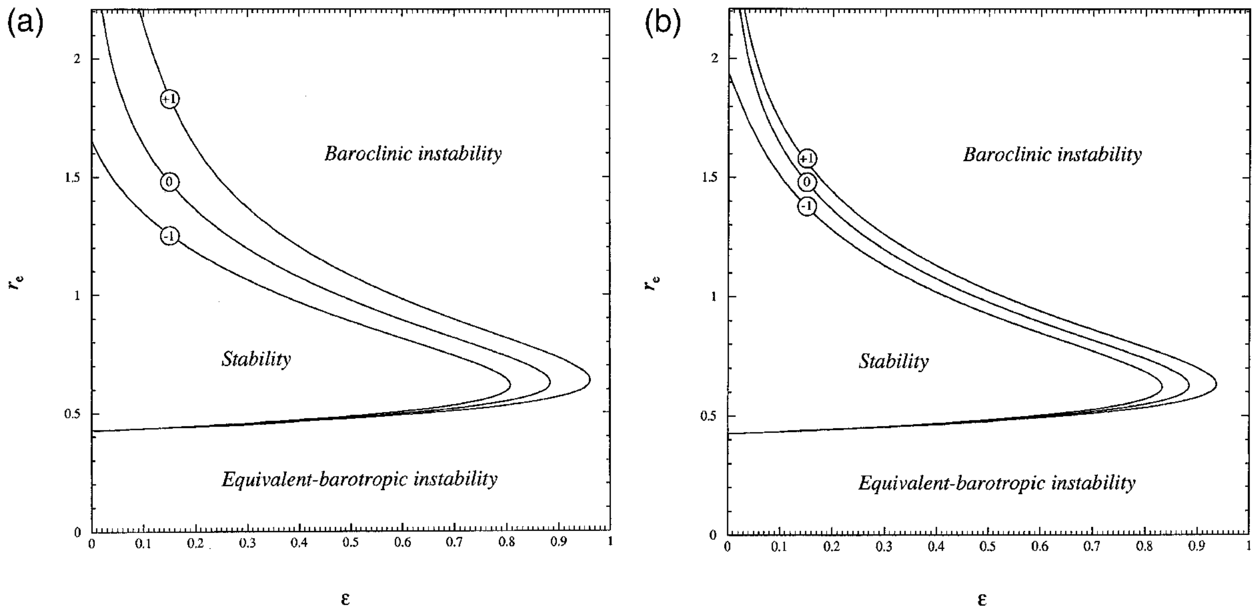

FIG. 1. The marginal stability curve for the Gaussian vortex over Gaussian topography, and the second azimuthal mode. Here ε is the depth ratio of the ocean and r_e is the radius of the vortex [(8)]. The topography parameter is $\delta = -0.1$, 0, and 0.1; the corresponding curves are marked with -1, 0, and 1, respectively. Curves are shown for (a) $r_t = 2$ (r_t is the radius of the topographic feature) and (b) $r_t = 1$.

where r_e is its nondimensional radius (i.e., the ratio of the dimensional radius to L_d). The topographic slope will also be assumed to be Gaussian,

$$S(r) = -\frac{r}{r_t}\exp\left(-\frac{r^2}{2r_t^2}\right), \tag{9}$$

where r_t is its nondimensional radius.[1] Recalling that S is the bottom slope, we obtain the following expression for the profile of topography:

$$D(r) = r_t \exp\left(-\frac{r^2}{2r_t^2}\right). \tag{10}$$

For the vortex in (8) and the topography in (9), the eigenvalue problem (5)–(7) was solved numerically using a technique described by Benilov (2003). Various azimuthal modes were examined, and, in most cases, the second one ($k = 2$) turned out to be the most unstable. The marginal stability curve of this mode on the (ε, r_e) plane, for various δ and r_t, is shown in Fig. 1.

The following tendencies can be observed:

1) Bottom elevations stabilize cyclones and bottom depressions stabilize anticyclones (in both cases, $\delta > 0$)—see Figs. 1 and 2a,b.
2) Vice versa, depressions destabilize cyclones and elevations destabilize anticyclones (in both cases, $\delta < 0$)—see Figs. 1 and 2c,d.
3) The stability properties of the vortex are very sensitive to the radius of the topographic feature: a wider feature affects them more strongly than a narrow one—compare Figs. 1a and 1b. However, as we shall see later, there is an "optimal" width for which the stabilizing effect of topography is strongest (after all, an infinitely wide elevation is equivalent to the flat-bottom case).
4) The most interesting (oceanographically) case of a thin upper layer is much more sensitive to topography than are cases of comparable layers—see the region $\varepsilon \ll 1$ in Figs. 1a,b.

Note also that, for small r_e, instability is caused by strong horizontal shear, which is why the corresponding region in Fig. 1 is labeled "equivalent-barotropic instability." At large r_e, horizontal shear is weak and instability is caused by vertical shear, which is why the corresponding region in Fig. 1 is labeled "baroclinic instability."

Note that our conclusions are opposite to those of Nycander and Lacasce (2004), who found that bottom elevations stabilize anticyclones, not cyclones. This discrepancy, however, does not cause a paradox, because Nycander and Lacasce (2004) examine *barotropic* instability, whereas we mostly consider the *baroclinic* one.

We have also calculated the threshold values of δ that would stabilize a "typical" vortex—one with parameters derived from Olson's (1991) data. Averaging the parameters of the eddies cataloged by Olson, we obtain

[1] Observe that V_1 and S are both positive; that is, the signs of the corresponding dimensional quantities are incorporated into the sign of δ. For example, the combination of a cyclone ($U_{*1} > 0$) and a sea mountain ($\Delta_{*2} < 0$) corresponds to $\delta < 0$ [see (3)].

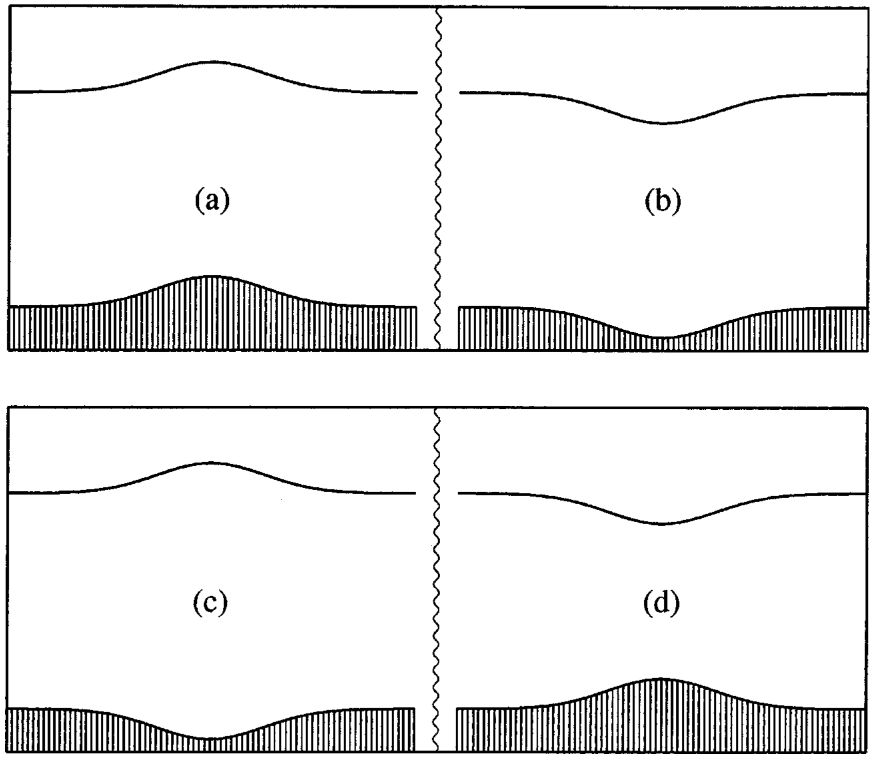

FIG. 2. Eddies over topography: (top) cases in which topography is a stabilizing influence and (bottom) those in which it is a destabilizing one.

the following typical value for the nondimensional radius of the eddy:

$$r_e \approx 2.47. \tag{11}$$

The thicknesses of the eddies are not presented by Olson (1991); thus ε had to be chosen on a more or less ad hoc basis:

$$\varepsilon = 0.075 \tag{12}$$

[this value has been previously used by Benilov (2003, 2004)]. For parameters (11) and (12), the threshold value of δ for which the eddy becomes stable is

$$\begin{aligned} &\delta \approx 0.1561 \quad \text{for } r_t = 2, \\ &\delta \approx 0.0645 \quad \text{for } r_t = 3, \\ &\delta \approx 0.0247 \quad \text{for } r_t = 4, \quad \text{and} \\ &\delta \approx 0.0261 \quad \text{for } r_t = 5. \end{aligned} \tag{13}$$

Among other things, these results suggest that the optimal value of r_t is somewhere in the region of

$$4 < r_t < 5.$$

In comparing the optimal r_t with the vortex's radius (11), we conclude that the former is approximately 2 times the latter.

To put results (13) into oceanographic context, observe that (8) and (10) correspond to

$$\max[V_1(r)] = e^{-1/2} \quad \text{and} \quad \max[D(r)] = r_t.$$

These equalities, in turn, imply that the velocity and topography are nondimensionalized by

$$U_{*1} = e^{1/2} V_{*1\,\max} \quad \text{and} \quad \Delta_{*2} = r_t^{-1} \Delta H_{*2},$$

respectively, where $V_{*1\max}$ is the maximum velocity and ΔH_{*2} is the height of the topographic feature (recall that the asterisks denote dimensional quantities). Substituting U_{*1} and Δ_{*2} into the definition of δ [see (3)] and rearranging it, we obtain

$$\frac{\Delta H_{*2}}{\overline{H}_{*2}} = e^{1/2} \delta r_t r_e \mathrm{Ro}, \tag{14}$$

where

$$\mathrm{Ro} = \frac{V_{*1\,\max}}{(L_d r_e) f_0} \tag{15}$$

is the Rossby number. For the eddies listed by Olson (1991), however, a different Rossby number is presented—the one based on the maximum *three-dimensional* velocity—whereas (15) involves the maximum *upper-layer* velocity (this is the price we have to pay for using the two-layer model). We assume that the latter is one-half of the former (which implies an assumption that the velocity in the "active" layer decays with depth linearly). Then, averaging Olson's data, we obtain the following Rossby number of a typical two-layer vortex:

$$\mathrm{Ro} \approx 0.105. \tag{16}$$

Substituting (11), (13), and (16) into (14), we obtain

$$\begin{aligned}
&\frac{\Delta H_{*2}}{\overline{H}_{*2}} \approx 0.133 \quad \text{for } r_t = 2,\\
&\frac{\Delta H_{*2}}{\overline{H}_{*2}} \approx 0.083 \quad \text{for } r_t = 3,\\
&\frac{\Delta H_{*2}}{\overline{H}_{*2}} \approx 0.042 \quad \text{for } r_t = 4, \quad \text{and}\\
&\frac{\Delta H_{*2}}{\overline{H}_{*2}} \approx 0.056 \quad \text{for } r_t = 5.
\end{aligned} \tag{17}$$

Thus, oceanic eddies can be stabilized by relatively weak topography. To relate the above values of r_t to the real ocean, note that the mean deformation radius derived from Olson's (1991) data is $L_d \approx 27$ km. Hence, (17) corresponds to the dimensional radius of the topographic feature that is 60–120 km.

Last, we mention that other vortex profiles were examined and yielded results similar to those for the Gaussian vortex.

4. Asymptotic analysis

In this section, we present an asymptotic theory based on the assumption that the upper (active) layer of the ocean is much thinner than the lower (passive) layer; that is,

$$\varepsilon \ll 1.$$

This assumption is not very restrictive, because real oceanic eddies are indeed mostly thin. We also assume that the depth variation is weak,

$$\delta \ll 1,$$

which is more restrictive, because it eliminates from consideration continental shelf and midocean ridges.

Note that the stability of vortices in a two-layer ocean with a thin upper layer has been recently examined for the case of a flat bottom (Schecter et al. 2001; Schecter and Montgomery 2003; Benilov 2003). Observe also that, except for the topographic term (the one involving δ), our (5) and (6) are exactly the same as equations (3.1) and (3.2) of Benilov (2003). As a result, the asymptotic approach used by Benilov can be readily extended to the present case—by using this approach, an approximate stability criterion can be derived that will help us to explain the numerical results presented above.

a. Leading-order results: Classification of modes

Consider the eigenvalue problem (5)–(7) and assume that $\delta \sim \varepsilon$ (which, in fact, includes also the limits $\delta \ll \varepsilon$ and $\delta \gg \varepsilon$). Accordingly put

$$\delta = \varepsilon\hat{\delta},$$

where $\hat{\delta} = O(1)$, and expand the solution in powers of ε,

$$\phi_{1,2} = \phi_{1,2}^{(0)} + \varepsilon\phi_{1,2}^{(1)} + \cdots \quad \text{and} \quad c = c^{(0)} + \varepsilon c^{(1)} + \cdots.$$

To leading order, (5)–(7) yield

$$[c^{(0)}r - V_1]\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi_1^{(0)}}{dr}\right] - \frac{k^2}{r^2}\phi_1^{(0)} - \phi_1^{(0)} + \phi_2^{(0)}\right\}$$
$$+ \left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rV_1)\right] - V_1\right\}\phi_1^{(0)} = 0,$$
$$c^{(0)}r\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi_2^{(0)}}{dr}\right] - \frac{k^2}{r^2}\phi_2^{(0)}\right\} = 0, \quad \text{and}$$
$$\phi_{1,2}^{(0)}(0) = \phi_{1,2}^{(0)}(\infty) = 0.$$

The lower-layer problem suggests either $c^{(0)} = 0$, or $\phi_2^{(0)} = 0$, or both. In accord, we shall distinguish three types of modes.

1) If

$$\phi_2^{(0)} = 0 \quad \text{and } c^{(0)} \neq 0,$$

the upper-layer problem, to leading order, decouples from its lower-layer counterpart. It describes the usual equivalent-barotropic motion, and its solutions will be referred to as upper-layer-dominated (ULD) modes.

2) If

$$\phi_2^{(0)} \neq 0 \quad \text{and } c^{(0)} = 0,$$

the lower-layer problem decouples from its upper-layer counterpart:

$$c^{(1)}r\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi_2^{(0)}}{dr}\right] - \frac{k^2}{r^2}\phi_2^{(0)}\right\}$$
$$+ (V_1 + \hat{\delta}S)\phi_2^{(0)} = 0 \quad \text{and} \tag{18}$$
$$\phi_2^{(0)}(0) = \phi_2^{(0)}(\infty) = 0, \tag{19}$$

and determines the eigenvalue $c^{(1)}$. It describes oscillations in a layer of variable thickness, and its solutions will be referred to as lower-layer-dominated (LLD) modes.[2] They exist because of the curvature of the bottom and interface and, in dynamic terms, are not sensitive to the flow in the upper layer. The upper-layer problem:

[2] Although ϕ_1 and ϕ_2, in this case, are of the same order, the larger thickness of the lower layer makes it dominant.

$$-V_1\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi_1^{(0)}}{dr}\right]-\frac{k^2}{r^2}\phi_1^{(0)}-\phi_1^{(0)}+\phi_2^{(0)}\right\}$$
$$+\left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rV_1)\right]-V_1\right\}\phi_1^{(0)}=0 \quad \text{and} \tag{20}$$
$$\phi_1^{(0)}(0)=\phi_1^{(0)}(\infty)=0, \tag{21}$$

describes an oscillation forced by the lower-layer motion, $\phi_2^{(0)}$.

3) If

$$\phi_2^{(0)}=0 \quad \text{and} \quad c^{(0)}=0,$$

the eigenvalue c, to leading order, drops out from the upper-layer problem—which then becomes

$$-V_1\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi_1^{(0)}}{dr}\right]-\frac{k^2}{r^2}\phi_1^{(0)}-\phi_1^{(0)}\right\}$$
$$+\left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rV_1)\right]-V_1\right\}\phi_1^{(0)}=0 \quad \text{and} \tag{22}$$
$$\phi_1^{(0)}(0)=\phi_1^{(0)}(\infty)=0. \tag{23}$$

Then $c^{(1)}$ is to be determined from the lower-layer problem:

$$c^{(1)}r\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi_2^{(1)}}{dr}\right]-\frac{k^2}{r^2}\phi_2^{(1)}+\phi_1^{(0)}\right\}$$
$$+(V_1+\hat{\delta}S)\phi_2=0 \quad \text{and} \tag{24}$$
$$\phi_2^{(1)}(0)=\phi_2^{(1)}(\infty)=0, \tag{25}$$

which, however, involves both eigenfunctions, $\phi_1^{(0)}$ and $\phi_2^{(1)}$. Hence, the corresponding solutions will be referred to as mixed (M) modes.

The ULD (equivalent barotropic) modes are not sensitive to topography and, hence, are irrelevant to this paper.[3] Thus, in what follows, we shall be concerned with LLD and M modes only.

It can be demonstrated [see Benilov (2003), in which a similar problem has been examined] that

- the LLD-mode problem (18)–(21) has a solution only for $k = 2$;
- the M-mode problem (22)–(25) has a solution only for $k = 1$, in which case

$$\phi_1^{(0)}=V_1(r); \quad \text{and,} \tag{26}$$

- in either case, $\mathrm{Im}c^{(1)} = 0$.

Thus, modes of both types are, to leading order, stable.

[3] Moreover, as shown by Benilov (2003), real mesoscale eddies are stable with respect to ULD modes.

b. *Higher-order results: Critical levels*

Consider the so-called critical levels (radii)—that is, the points $r = r_c$, at which the angular phase speed of the disturbance equals the angular velocity of the fluid,

$$\mathrm{Re}c=\frac{1}{r_c}V_1(r_c).$$

For simplicity, we shall assume that $(1/r)V_1(r)$ is a monotonic function; hence, no more than a single critical level exists in the problem. For LLD and M modes (for which c scales with ε), it is located at the periphery of the vortex, where V_1 is small. Most important, the expansion developed above fails near $r = r_c$, where cr becomes comparable to V_1 and cannot be neglected [as has been done when we replaced (5) with (20) or (22)]. Note that, because the critical level did not arise in the leading order, its contribution to the eigenvalue c must be small—but no matter how small, it can be complex and, thus, cause instability.

The effect of critical levels can be taken into account using an asymptotic technique developed by Benilov (2003) for the case of flat bottom. The two cases are, in fact, so similar that we shall present here only the conclusions and refer the mathematically minded reader to Benilov's (2003) paper.

It turns out that the stability of an LLD or M mode depends on the position of its critical level—which, in turn, depends on the phase velocity $c^{(1)}$ calculated through the leading-order problem (18)–(19) or (22)–(25).

- If $c^{(1)}$ is such that no critical level exists [e.g., if $V_1(r)$ is a sign-definite function and $c^{(1)}$ is of the opposite sign], the disturbance is neutrally stable [i.e., the eigenvalue of the original problem (5)–(7) is real].
- If a critical level $r = r_c$ exists and the upper-layer velocity and PV gradient at $r = r_c$ are of opposite signs,

$$V_1(r_c)Q_1'(r_c)<0, \tag{27}$$

(5)–(7) have an unstable eigenvalue.
- If a critical level exists, but the upper-layer velocity and PV gradient at $r = r_c$ are of the same sign, no eigenvalue exists for (5)–(7).

c. *Discussion: How does topography affect the critical levels?*

As seen above, the exact eigenvalue problem (5)–(7) can be reduced to the leading-order problems (18)–(19) and (24)–(26). The former is a fourth-order set of ODEs with singular (at the critical level) coefficients, whereas the latter are of second order and have regular coefficients—which makes them easier to solve numerically.

Another advantage of the asymptotic approach is that it will enable us to interpret the results physically.

Consider, for example, LLD modes and observe that the eigenvalue problem (18)–(19) that describes them is invariant with respect to the transformation

$$V_1 + \hat{\delta}S \rightarrow a(V_1 + \hat{\delta}S) \quad \text{and} \quad c^{(1)} \rightarrow ac^{(1)},$$

where a is an arbitrary constant. Hence, assuming for simplicity that $V_1 > 0$, we note that a positive $\hat{\delta}S$ increases the phase speed $c^{(1)}$, whereas a negative one decreases it.

Next, observe that an increase in the phase velocity moves the critical level toward the vortex's core—see a schematic in Fig. 3a. Because the PV gradient typically is negative there (see Fig. 3b), this can "enforce" our instability criterion (27) and make the disturbance unstable. Vice versa, if V_1 and $\hat{\delta}S$ are of opposite signs, the latter causes a decrease in $c^{(1)}$, which, in turn, moves the critical level toward the periphery of the vortex and eventually stabilizes it.

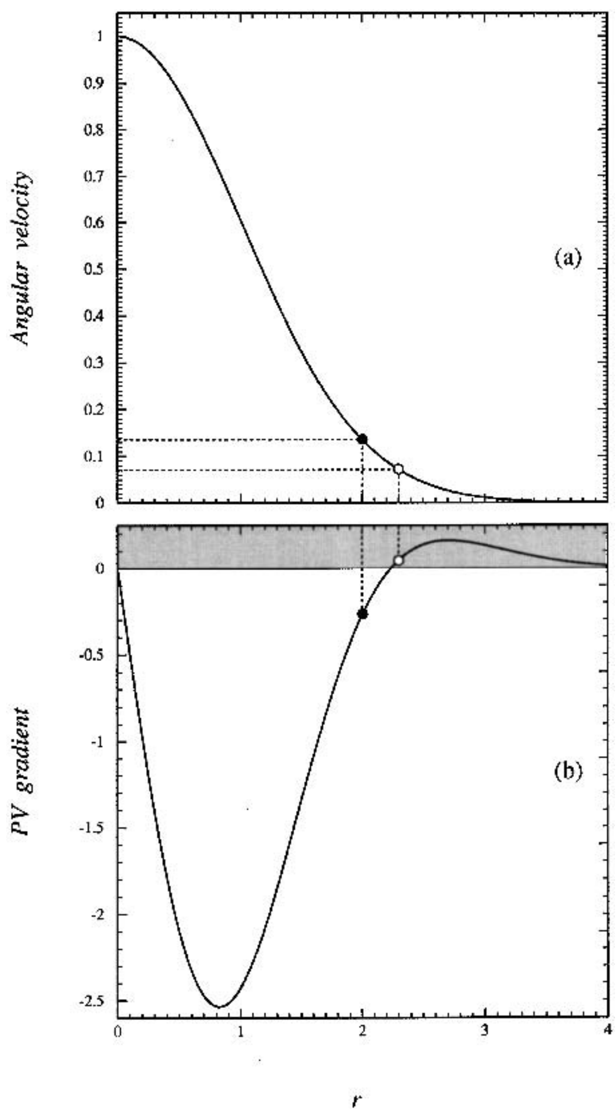

FIG. 3. The upper-layer (a) angular velocity $(1/r)V_1(r)$ and (b) PV gradient $Q_1'(r)$ for a Gaussian vortex. The region of positive Q_1' is shaded. Circles show the positions of the critical level: the open circle corresponds to $S = 0$ (flat bottom), the filled circle corresponds to the case in which S and V_1 are of the same sign. In the latter case, the critical level occurs in the region where V_1Q_1' < 0 (instability).

5. Summary and concluding remarks

Thus, we have examined the stability of a radially symmetric vortex over radially symmetric topography in a two-layer ocean. The main conclusion is that a bottom elevation is a stabilizing influence for cyclones and a destabilizing influence for anticyclones, whereas a depression acts in the opposite way (see Fig. 2).

Note also that the stability properties of a vortex are very sensitive to the radius of the topographic feature: a wider feature affects them more strongly than does a narrow one (unless it is much larger than the vortex, in which case it is equivalent to a flat bottom). It was also observed that the case of a thin upper layer, which is the most interesting oceanographically, is much more sensitive to topography than is the case of comparable layers.

The results obtained can be interpreted using an asymptotic theory developed for the case of a thin upper layer. It is demonstrated that a bottom elevation moves the critical level of an unstable mode toward the periphery of the cyclone, which leads to its stabilization. Estimates based on realistic oceanic parameters show that stabilization may occur for relatively small topography (5%–15% of the lower layer's depth).

Last, note that this study is only a first step in studying the stability of vortices over topography. To achieve clarity in this matter, the results should be generalized for *ageostrophic* vortices—such that the Rossby number is of order 1 while the displacement of the interface is comparable to the depth of the upper layer. Furthermore, as argued by Dewar and Killworth (1995) and Benilov (2004), oceanic eddies are often accompanied by a *deep flow*, which can dramatically affect their stability properties. For this reason, the comprehensive model of vortex stability should also include a (weak) circulation in the lower layer.

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