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Scattering of Internal Waves in a Current above a Nonuniform Bottom

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Scattering of a monochromatic internal wave by the random bottom irregularities in a stratified ocean and in the presence of currents with velocity shear is investigated. The Hamiltonian approach is used to derive an equation relating the amplitude of the scattered wave to time. The internal wave amplitude increases during scattering over a wide range of wave vectors.

1. Introduction

Studying the scattering of waves in dispersive media with fluctuating characteristics is an important problem of modern wave theory. There have been numerous publications dealing with this subject in recent years (see, e.g., [1 - 4] and the literature cited in them). Nonetheless, no complete solution of what we regard as the central problem, whether wave amplitudes can be increased by scattering, has yet been obtained. For example, Naugol'nykh and Rybak [1] have demonstrated that if fluctuations in the characteristics of the medium result from the presence of a random high-frequency wave field, then such growth is possible if certain additional conditions are satisfied. But the interaction of waves with fluctuating, comparable time and space scales is of great interest, and wave scattering by stationary (and not only "wavelike") spatial inhomogeneities in the medium is important in real physical situations. For example, spatially nonuniform currents and bottom irregularities are present everywhere in the ocean, and internal waves scattered by them will always exchange energy effectively. Scattering of internal waves by random bottom irregularities in an immobile ocean is studied by Yermakov [2] (see also Pelinovskiy's survey [3]). In this case, the amplitude of the scattered wave proves to decrease over time regardless of the bottom relief structure or the other ocean characteristics.

In this paper we investigate scattering of a monochromatic internal wave in a stratified ocean with a randomly irregular bottom in the presence of mean flow. It should be noted that in this situation, an increase in the wave amplitude is expected. Indeed, in the presence of flow, internal waves carrying negative energy arise over a wide range of wave numbers (i.e., the "retardation" of mean flow, see [5]). When these interact (by mutual scattering) with internal waves of positive energy, they will increase while leaving the total energy of the internal wave field constant [5, 6]. Note also that the simultaneous presence of bottom irregularities and mean flow is typical of the ocean shelf, and this effect will thus be significant in the energetics of internal waves propagating in the seasonal thermocline.

2. Formulation of the Problem

Consider a plane-parallel current $U(z)$ in a stratified ocean with an irregular bottom specified by the equation $z = s(\mathbf{r})$, where z and $\mathbf{r} = (x, y)$ are the vertical and horizontal coordinates (the surface $z = 0$ represents the mean depth of the bottom). We shall consider small bottom irregularities ($|h| \ll H$, where H is the average depth of the ocean); in this case, the wave field can be represented as a superposition of internal waves with different wave vectors and amplitude. To simplify the calculations, we consider the simplest, two-layered, model of ocean stratification, which allows of only one internal wave, and we shall demonstrate at the end how the results can be extended to the multimode case.

We use the Hamiltonian approach to this problem [7, 8], and, in application to a similar problem [4], which simplifies calculations and makes it possible to extend the results to waves in an arbitrary spatially nonuniform dispersive medium (e.g., surface waves or Rossby waves in an ocean with an irregular bottom). We therefore introduce the so-called "complex normal internal wave amplitude" [7, 8] $a_{\mathbf{k}}$, where $\mathbf{k} = (k_x, k_y)$ is the wave vector (the relationship of $a_{\mathbf{k}}$ to the natural hydrodynamic variables is derived in [8]). We represent the energy of the wave field (the Hamiltonian) as an integral-power series in the parameter h/H and limit ourselves to the first three terms,

$$\mathcal{H} = \int \omega_k |a_k|^2 dk + \iiint \Big[U_{kk_1} a_k^* a_{k_1} h_{k_2} \delta(k - k_1 - k_2) + + \frac{1}{2} (U_{kk_1}^* a_k a_{k_1}^* h_{k_2}^* + V_{kk_1} a_k a_{k_1} h_{k_2}) \delta(k + k_1 + k_2) \Big] dk\, dk_1\, dk_2. \quad (1)$$

Here $\omega = \omega_k$ is the dispersion dependence of internal waves in the current, h_k is the Fourier transformation of the function $h(r)$; $\delta(k)$ is the Dirac delta function, U_{kk_1} and V_{kk_1} are the matrix elements giving the probability that a wave with a wave vector k is scattered into a wave with wave vector k_1. Omitting the extremely cumbersome expressions for U_{kk_1} and V_{kk_1} in terms of the eigenfunctions of the corresponding boundary value problem, we simply note that all of the results obtained above are independent of their particular form. The matrix elements for the specific case of the absence of bottom flow $(U(z) \equiv 0$ for $z \leqslant \max\{h(r)\})$ are calculated by Benilov and Chernyak [5]. The only equations of interest to us are those

$$U_{k_1k} = U_{kk_1}^*, \quad V_{k_1k} = V_{kk_1}, \quad (2)$$

which ensure that the Hamiltonian is real and symmetrical. To simplify the technical aspect of the problem, we discard from Eq. (1) the terms representing nonlinear wave interaction (i.e., assuming small internal wave amplitudes).

The dynamic equation for a_k is $\dot{a}_k = i\delta\mathcal{H}/\delta a_k^*$, where $\dot{a}_k \equiv \partial a_k/\partial t$, and $\delta\mathcal{H}/\delta a_k^*$ denotes the variational derivative. By the variational procedure we obtain

$$\dot{a}_k = i\omega_k a_k + i \iint [U_{kk_1} a_{k_1} h_{k_2} \delta(k - k_1 - k_2) + V_{kk_1} a_{k_1}^* h_{k_2}^* \delta(k + k_1 + k_2)]\, dk_1\, dk_2. \quad (3)$$

Note that Eq. (3) then describes the evolution of waves in an arbitrary space-time system, depending on the specific form of the coefficients ω_k, U_{kk_1} and V_{kk_1}, for example, the problem of scattering of barotropic Rossby waves by bottom inhomogeneities can also be reduced to an equation of the form (3) [9]. Thus, all of the results obtained above will be formulated in such a way that they can readily be generalized to dispersive waves of arbitrary type.

3. Evolution of the Amplitude of a Monochromatic Wave in a Random Medium

Consider a monotonic internal wave with a wave vector k = ϰ, a frequency $\omega = \omega_\varkappa$, and amplitude $A(t)$. The solution of Eq. (3) that describes its evolution will also contain a continuous spectrum of stochastic waves scattered by random bottom irregularities, $b_k(t)$, and a coherent monochromatic reflected wave $(k = -\varkappa,\ \omega = \omega_{-\varkappa}$; $B(t)$ is the amplitude):

$$a_k = A\delta(k - \varkappa) + b_k + B\delta(k + \varkappa). \quad (4)$$

Now, substituting Eq. (4) into Eq. (3) and averaging it,

$$(\dot{A} - i\omega_\varkappa A)\,\delta(k - \varkappa) + (\dot{B} - i\omega_{-\varkappa} B)\,\delta(k + \varkappa) = = i \iint [U_{kk_1} \langle b_{k_1} h_{k_2} \rangle \delta(k - k_1 - k_2) + V_{kk_1} \langle b_{k_1}^* h_{k_2}^* \rangle \delta(k + k_1 + k_2)]\, dk_1\, dk_2, \quad (5)$$

where $\langle h_k \rangle = \langle b_k \rangle = 0$, and the angle brackets denote averaging over the ensemble of realizations. It is evident that the scattered internal wave with vector k_1 will be correlated with the spectral component $h_{k_2}|_{k_2 = k_1 \mp \varkappa}$ (since both are involved in the scattering of coherent waves A and B). This fact leads to the appearance of so-called "anomalous" correlators

$$\langle b_{k_1} h_{k_2} \rangle = J_{k_1} \delta(k_1 - \varkappa + k_2) + I_{k_1} \delta(k_1 + \varkappa + k_2). \quad (6)$$

Substituting Eq. (6) into Eq. (5) and separating the equations for A and B, we obtain

$$\dot{A} - i\omega_\varkappa A = i \int (U_{\varkappa k_1} J_{k_1} + V_{\varkappa k_1} I_{k_1}^*)\, dk_1,$$
$$\dot{B} - i\omega_{-\varkappa} B = i \int (U_{-\varkappa k_1} I_{k_1} + V_{-\varkappa k_1} J_{k_1}^*)\, dk_1. \quad (7)$$

The evolutionary equations for J_k and I_k that close system (7) are obtained by multiplying the original equation (3) by h_{k_3} and averaging over the ensemble of realizations. Assuming that the amplitude of the scattered wave is compared with $A \left(\left| \iint \langle b_k b_{k_1}^* \rangle dk\, dk_1 \right| \ll |A|^2 \right)$, we discared the terms $\sim \langle b_{k_1} h_{k_2} h_{k_3} \rangle$ from the resulting averaged equation. We thus assume single scattering (the Bourre approximation or modified Born approximation), which is valid in this case by virtue of the bottom irregularities, and which has been utilized previously. We obtain

$$\dot{J}_k - i\omega_k J_k = i(U_{k\varkappa} A + V_{k,-\varkappa} B^*) N_{k-\varkappa},$$
$$\dot{I}_k - i\omega_k I_k = i(U_{k,-\varkappa} B + V_{k\varkappa} A^*) N_{k+\varkappa}, \quad (8)$$

where N_k is defined by the equation $\langle h_k^* h_{k_1}^* \rangle = N_k \delta(k - k_1)$ $(N_{-k} = N_k)$. We change over in system of equations (7), (8) to the new dependent variables by means of the equations $A = \tilde{A} e^{i\omega_\varkappa t}$, $B = \tilde{B} e^{i\omega_{-\varkappa} t}$, $J_k = \tilde{J}_k e^{i\omega_k t}$, $I_k = \tilde{I}_k e^{i\omega_k t}$ and introduce an infinitesimal wave damping in the equations for J_k and I_k (where $\omega_k \to \omega_k + i0$), after which we integrate them with initial conditions $\tilde{J}_k|_{t \to -\infty} = \tilde{I}_k|_{t \to -\infty} = 0$. For $\tilde{J}_k$, for example, we obtain

$$\tilde{J}_k = iN_{k-\varkappa} \Big\{ U_{k\varkappa} \int_{-\infty}^{t} \tilde{A}(t') \exp[i(\omega_\varkappa - \omega_k - i0)t']\, dt' + + V_{k,-\varkappa} \int_{-\infty}^{t} \tilde{B}^*(t') \exp[i(-\omega_{-\varkappa} - \omega_k - i0)t']\, dt' \Big\}. \quad (9)$$

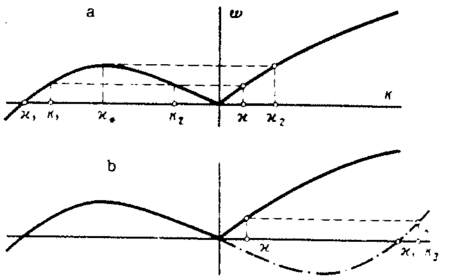

Fig. 1

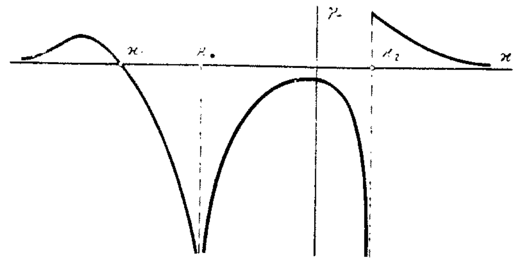

Fig. 2

Fig. 1. Dispersion curves for internal waves in flow, for planar problem ($\varkappa$ and k parallel to U): a) resonance of type (14a); b) resonance of type (14b). Dashed-dotted line is curve for symmetrical reflection of left branch of the dispersion plot $\omega=\omega_{-k}$.

Fig. 2. Qualitative behavior of coefficient γ_+ as function of wave vector of scattered wave $\varkappa$. Wave amplitude increases for $\gamma_+ > 0$.

The internal wave damping introduced above causes the integrals on the right side of (9) to converge. In addition, since we have been considering weak scattering from the outset ($|h| \ll H$), the characteristic time T over which the amplitudes of waves A and B change is much greater than their periods, ($T \gg 1/|\omega_{\pm\varkappa}|$), and we can calculate J_k for large times by means of an asymptotic formula that is valid for $t \gg 1/|\omega_{\pm\varkappa}|$:

$$\int_{-\infty}^{t} A(t')\exp[i(\omega_\varkappa - \omega_k - i0)t']\,dt' \approx A(t)\,\pi\delta(\omega_k - \omega_\varkappa) \quad (10)$$

this formula is proved in the Appendix. Using an analogous formula for the second integral in Eq. (9), we obtain

$$\tilde{J}_k = i\pi N_{k-\varkappa}[U_{k,\varkappa}A\delta(\omega_k - \omega_\varkappa) + V_{k,-\varkappa}B^*\delta(\omega_k + \omega_{-\varkappa})]. \quad (11a)$$

Repeating all of the calculations, we obtain for I_k

$$\tilde{I}_k = i\pi N_{k+\varkappa}[U_{k,-\varkappa}B\delta(\omega_k - \omega_{-\varkappa}) + V_{k,\varkappa}A^*\delta(\omega_k + \omega_\varkappa)]. \quad (11b)$$

We substitute Eqs. (11) into the equations for $\tilde{A}$ and $\tilde{B}$ and neglect the rapidly oscillating terms $\sim e^{\pm i(\omega_\varkappa \pm \omega_{-\varkappa})}$ (which can be justified by a strict asymptotic analysis of Eq. (7). After substituting Eq. (2), the resulting system of equations for $\tilde{A}$ and $\tilde{B}$ is separated and takes the form

$$\dot{\tilde{A}} = \gamma_+\tilde{A}, \quad \dot{\tilde{B}} = \gamma_-\tilde{B}, \quad (12)$$

$$\gamma_\pm = \pi\int[-|U_{\pm\varkappa,k}|^2 N_{k\mp\varkappa}\delta(\omega_k - \omega_{\pm\varkappa}) + |V_{\pm\varkappa,k}|^2 N_{k\pm\varkappa}\delta(\omega_k + \omega_{\pm\varkappa})]\,dk. \quad (13)$$

Thus, to a first-order approximation in this theory of perturbations, the amplitude of reflected wave B can be assumed to be zero, and the behavior of the amplitude of the incident internal wave depends on the sign of the coefficient γ_+ (for $\gamma_+ > 0$, the amplitude of A increases as $t \to \infty$).

4. Analysis of Eq. (13)

Note first that the two terms in Eq. (13) express two types of resonance between the incident internal wave ($\varkappa, \omega_\varkappa$) scattered by the harmonic (k, ω_k) and the spectral component of the bottom irregularities with wave vector k_1 and a frequency of zero:

$$k - \varkappa - k_1 = 0, \quad \omega_k - \omega_\varkappa = 0, \quad (14a)$$

$$k + \varkappa + k_1 = 0, \quad \omega_k + \omega_\varkappa = 0. \quad (14b)$$

Equation (14b) describes scattering of the "decay interaction" type (Fig. 1a), while (14b) describes the "explosive interaction" type* (Fig. 1b). Returning to the wave energy equation (1), we see that the energy density in the internal wave is proportional to its frequency. Thus, the first term in Eq. (13) describes scattering of internal waves into waves with the energy of the same sign, while the second describes the same process for energy of the opposite sign. We also note that the absence of mean flow is represented by the equality ($\omega_k \geqslant 0$), and Eq. (13) is identical, except for the notation, to the expressions derived in [3, 4] for an immobile ocean. All internal waves have positive energy ($\gamma_+ < 0$) and the law of conservation of energy forbids an increase

*This terminology is derived from nonlinear wave mechanics, which uses the same concepts.

in their amplitude ($\gamma_+ < 0$). The situation differs in the presence of flow: in this case the frequency of the internal wave is variable in sign (see Fig. 1) as a result of Doppler shift ($\omega_k \to \omega_k + (\mathbf{k}, \mathbf{U})$, for slow without vertical shear), both resonances of Eq. (14) are resolved, and the growth or decrease in the amplitude of the incident wave depends on their competition. Considering the planar problem for simplicity (with two spatial variables x and z), we determine the ranges of the characteristics at which the amplitude of the incident internal wave increases. Now the integrals in Eq. (13) are amenable to evaluation and can be converted to the form

$$\gamma_+ = \pi\left[- \sum_{i=1,2} |U_{\varkappa k_i}|^2 N_{k_i-\varkappa} |c_g(k_i)|^{-1} + |V_{\varkappa k_3}|^2 N_{k_3+\varkappa} |c_g(k_3)|^{-1}\right], \tag{15}$$

where $c_g(k) = d\omega_k/dk$ is the group velocity of the internal waves in the current, $k_{1,2}$ and k_3 are the roots of equations $\omega_k - \omega_\varkappa = 0$ and $\omega_k + \omega_\varkappa = 0$, respectively (see Fig. 1). Resonance (14a) is forbidden for $\varkappa < \varkappa_1$ and $\varkappa > \varkappa_2$ (Fig. 1a), the negative terms in (15) disappear, $\gamma_+ > 0$, and the amplitude of the incident wave increases. But with a further increase in the absolute value of the wave vector ($|\varkappa| \to \infty$), the internal wave begins to be increasingly insensitive to the bottom ($U_{\varkappa k_i}$, $V_{\varkappa k_i} \to 0$) and its growth increment γ_+ decreases exponentially. Nonetheless, for a shallow ocean ($H \sim 200 - 300$ m) and moderate flow velocities ($U \sim 20 - 50$ cm/sec), boundary conditions $k_{1,2}$ represent internal waves with $\lambda \sim 100 - 500$ m, which have good sensitivity to bottom inhomogeneities ($\lambda \sim H$) and this effect (the growth of internal waves during scattering) will be quite considerable. Note also that the first term in Eq. (14) increases without bound ($c_g(k_{1,2}) \to 0$) (Fig. 2) as $\varkappa \to \varkappa_2 - 0$ or $\varkappa \to \varkappa_* \pm 0$ (Fig. 1). We emphasize that the appearance of singularities of γ_+ at these points is a consequence of the inconsistent applicability of our theory of perturbations: it is inapplicable in their small neighborhoods. Nonetheless, the qualitative trend (the presence of minima of γ_+ for $\varkappa = \varkappa_2, \varkappa_*$) is valid.

Generalizing the above, we note that all specific characteristics of this wave medium prove to be included in the dispersion equation for internal waves in a flow, $\omega = \omega_k$. But the form of the matrix elements U_{kk_1} and V_{kk_1} affects only the absolute value (and not the sign) of the coefficient γ_+. The estimate [5] $U_{\varkappa k} \sim V_{\varkappa k} \sim (\omega_\varkappa / \varkappa k H^3) e^{-(k+\varkappa)H}$ is correct as to order of magnitude, and we can write $N \sim \langle h^2 \rangle R$, for N_k in the two-dimensional case, where $\langle h^2 \rangle$ is the rms value (variance) of the function $h(x)$, and R is its correlation radius. Consider now the case in which the range of interaction is not shifted into the short-wave region ($\varkappa \sim k \sim 1/H$); we obtain

$$\gamma_+ \sim \omega_\varkappa \frac{\langle h^2 \rangle}{H^2} \frac{R}{H}. \tag{16}$$

As can be seen from Eq. (16), the actual small parameter in this perturbation theory, when the logical condition $R \sim H$ is satisfied, is $(h/H)^2$ (rather than h/H)*. Therefore, substituting $\langle h^2 \rangle / H \sim 0.1$ and $\omega_\varkappa \sim 0.1 - 0.05$ min^{-1}, we find that the characteristic time scale for the increase in the scattered wave is $T \sim 1/\gamma_- \sim 1.5 - 3.5$ hours.

In conclusion, we present the formula generalizing (13) for the multimode case (i.e., for continuous stratification models):

$$\gamma_+ = \pi \sum_\nu \int [-|U^{\mu\nu}_{\varkappa k}|^2 N_{k-\varkappa} \delta(\omega^\nu_k - \omega^\mu_\varkappa) + |V^{\mu\nu}_{\varkappa k}|^2 N_{k+\varkappa} \delta(\omega^\nu_k + \omega^\mu_\varkappa)]\, dk, \tag{17}$$

where ν is the mode number (μ is the mode number of the incident wave). Obviously, Eqs. (13) and (17) do not differ from each other in any fundamental way and our estimate remains valid.

APPENDIX

Proof of Eq. (10)

Introducing the new notation $\Omega = \omega_\varkappa - \omega_k$, we rewrite the left side of Eq. (10) as

$$F = \lim_{\theta \to +0} \int_{-\infty}^{t} \tilde{A}(t') e^{(i\Omega + \theta)t'} dt'.$$

As noted earlier, the characteristic time scale of changes in $A(t)$ is much greater than $1/\Omega$ and the stationary phase method can be used to calculate F. Since stationary points are absent, the tails of the integrand [10] make the principal contribution to F; integrating by parts, we therefore obtain

$$F = A(t) \frac{e^{i\Omega t}}{i\Omega} - \dot{A}(t) \frac{e^{i\Omega t}}{(i\Omega)^2} + \ldots. \tag{A1}$$

We are interested in the asymptotic behavior of S for long time periods. Consequently, using the equation $\lim_{t \to \infty} \frac{e^{i\Omega t}}{i\Omega} = \pi\delta(\Omega)$ (see [11]) and retaining in expansion (A.1) only the first term, we obtain the desired expression for F.

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*It is convenient to determine the range of applicability of these results *a posteriori* from the condition $\gamma_- \ll |\omega_\varkappa|$. It can also be demonstrated that all of the subsequent corrections to the frequency of the scattered internal waves $\omega_\varkappa$ will be proportional to even powers of the parameter h/H.

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