1. Linear State Equations

   The LTI continuous-time equation $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0$
   has solution
   $$\mathbf{x}(t) = e^{A(t-t_0)}\{\mathbf{x}_0 + \int_{t_0}^{t} e^{A(t_0-\tau)}B\mathbf{u}(\tau)\,d\tau\}$$

2. Computing A^k and e^{At} when A is diagonalisable ($A = E\Lambda E^{-1}$)
   $$A^k = E\Lambda^k E^{-1}$$
   $$e^{At} = E\left(\sum_{j=0}^{\infty}\frac{\Lambda^j t^j}{j!}\right)E^{-1} = Ee^{\Lambda t}E^{-1}$$

3. *Lyapunov*'s Direct Method for Linear systems
   The (continuous-time) *Lyapunov* Equation is
   $$A^TP + PA = -Q$$

4. Transformations
   The transformation $\mathbf{x} = T\mathbf{z}$ transforms $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u},\ \mathbf{y} = C\mathbf{x}$ to
   $\dot{\mathbf{z}} = T^{-1}AT\mathbf{z} + T^{-1}B\mathbf{u},\ \mathbf{y} = CT\mathbf{x}$.

5. Controllability, Observability, & Duality

   (a) The Controllability U and Observability V matrices are given by:
   $$U = \begin{pmatrix} B, & AB, & A^2B, & \ldots, & A^{n-1}B \end{pmatrix} \qquad V = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

   (b) *Popov-Belevitch-Hautus* (PBH) test for controllability:
   $$M \triangleq [\lambda I - A, B]$$
   is of rank n for all λ if and only if (A, B) is completely controllable.

   (c) The PBH test for observability:
   $$N \triangleq \begin{bmatrix} C \\ \lambda I - A \end{bmatrix}$$
   is of rank n for all λ if and only if (A, C) is completely observable.

(d) Controllable Canonical Form for a Single Input System with characteristic polynomial $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$:

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

(e) The system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u},\ \mathbf{y} = C\mathbf{x}$
has *dual* $\dot{\mathbf{x}}^* = A^T\mathbf{x}^* + C^T\mathbf{u}^*,\ \mathbf{y}^* = B^T\mathbf{x}^*$

6. Pole Placement for Single Input Systems

(a) Transformation to ccf approach.

The transformation $\hat{T} = U\hat{U}^{-1}$, where U and $\hat{U}$ are the controllability matrices of the original and transformed systems respectively, transforms a completely controllable system into controllable canonical form.
The matrix $K = \hat{K}\hat{T}^{-1}$ where $\hat{k}_i = a_{i-1} - \hat{a}_{i-1}, \quad i = 1, 2, \ldots, n$ is the required feedback matrix.

(b) *Ackermann*'s Formula
The formula is

$$K = -[0, 0, \ldots, 0, 1]U^{-1}\chi_{cl}^{des}(A)$$

where $U = [b, Ab, \ldots, A^{n-1}b]$ is the controllability matrix of the open-loop system, and $\chi_{cl}^{des}(A) = A^n + \hat{a}_{n-1}A^{n-1} + \cdots + \hat{a}_1 A + \hat{a}_0 I$ is the desired characteristic polynomial of the closed-loop system matrix evaluated at the open-loop system matrix.

7. *Luenberger* Observers
The continuous-time full order *Luenberger* observer is

$$\dot{\mathbf{z}} = (A + LC)\mathbf{z} + B\mathbf{u} - L\mathbf{y}$$

8. Decoupling/ Non-interacting control
The decoupling index for output y_i is the smallest nonnegative integer d_i such that $c_i A^{d_i} B \neq 0$. Then

$$\hat{\mathbf{y}} = \begin{pmatrix} c_1 A^{d_1+1} \\ c_2 A^{d_2+1} \\ \vdots \\ c_m A^{d_m+1} \end{pmatrix} \mathbf{x} + \begin{pmatrix} c_1 A^{d_1} B \\ c_2 A^{d_2} B \\ \vdots \\ c_m A^{d_m} B \end{pmatrix} \mathbf{u}$$

where $\hat{y}_i = \frac{d^{d_i+1} y_i}{dt^{d_i+1}}$.The *remnant*, if it exists, is of order $q = n - \sum (d_i + 1)$.

9. Characteristic Polynomials and Stability

(a) The 2nd degree polynomial

$$\lambda^2 + a_1\lambda + a_0$$

has all its roots (eigenvalues, poles) in the left half plane (LHP) iff

$$a_1 > 0, \qquad a_0 > 0.$$

(b) The 3rd degree polynomial

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

has all its roots (eigenvalues, poles) in the LHP iff

$$a_2 > 0, \;\; a_1 > 0, \;\; a_0 > 0 \qquad \text{and} \qquad a_2 a_1 > a_0.$$

10. *Hamilton-Jacobi-Bellman* (HJB) Equation for infinite horizon problems.
The stabilising control that minises $J = \int_{t_0}^{\infty} L(\mathbf{x}, \mathbf{u})\, dt$ for the system
$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$ is given by the solution of

$$0 = \min_{\mathbf{u}(t)} \{ L(\mathbf{x}, \mathbf{u}) + \left(\frac{\partial V}{\partial \mathbf{x}} \right)^T \mathbf{f}(\mathbf{x}, \mathbf{u}) \}$$

where $V(\mathbf{x}) = \min_{\mathbf{u}(.)} J$.

11. Linear Quadratic (LQ) control
The minimum value of

$$J(\mathbf{u}) = \mathbf{x}^T(t_f) S \mathbf{x}(t_f) + \int_{t_0}^{t_f} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u}\, dt$$

for the system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \qquad\qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by $V(\mathbf{x}_0, t_0)$ where $V(\mathbf{x}, t) = \mathbf{x}^T P \mathbf{x}$ and $P = P^T$ is the solution of the matrix *Riccati* equation (MRE)

$$-\dot{P} = Q + A^T P + PA - PBR^{-1}B^T P \qquad\qquad P(t_f) = S$$

Furthermore $\mathbf{u}(t) = -R^{-1}B^T P \mathbf{x}(t)$.
Additionally, if $t_f \to \infty$, then for the LTI system with (A, B) completely controllable and (A, C) completely observable where $C^T C = Q$, P is the positive definite solution of the continuous-time algebraic *Riccati* equation (CARE)

$$0 = Q + A^T P + PA - PBR^{-1}B^T P$$

12. A modified *LaSalle* Invariance Principle: For the system

$$\dot{x} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

if (i) V is positive definite, (ii) $\frac{dV}{dt}$ is negative semi-definite and (iii) the union of all invariant sets with $\frac{dV}{dt} = 0$ is $M = \{\mathbf{0}\}$, then $\mathbf{0}$ is asymptotically stable.
In particular for the systems

$$\ddot{x} + b(\dot{x}) + c(x) = 0,$$

and

$$\ddot{x} + p(x)\dot{x} + c(x) = 0,$$

where b and c are continuous functions with the same sign as their arguments and $p(x) > 0$ then the origin $\mathbf{0}$ is asymptotically stable. This may be shown using

$$V(\mathbf{x}) = \frac{1}{2}x_2^2 + \int_0^{x_1} c(s)\, ds$$

13. Control *Lyapunov* Functions
    $V(\mathbf{x})$ is a control *Lyapunov* function for the affine control system

$$\dot{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

    if (i) it is positive definite and (ii)

$$b(\mathbf{x}) = \mathbf{0} \qquad \Rightarrow \qquad a(\mathbf{x}) < 0, \;\; \mathbf{x} \neq \mathbf{0}$$

    where

$$\frac{dV}{dt} = \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \dot{\mathbf{x}} = \underbrace{\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{f}(\mathbf{x})}_{a(\mathbf{x})} + \underbrace{\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{g}(\mathbf{x})}_{b(\mathbf{x})}\, \mathbf{u}$$

    Artstein's Theorem: An affine control system has a stabilising feedback control iff it has a control *Lyapunov* function.
    Sontag's formula: for a single input system a stabilising control is

$$u = k(\mathbf{x}) \triangleq \begin{cases} -\frac{a(\mathbf{x}) + \sqrt{a^2(\mathbf{x}) + b^4(\mathbf{x})}}{b(\mathbf{x})}, & \text{if } b(\mathbf{x}) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

14. Geometric Control

    (a) Differential Geometry notations:

$$L_{\mathbf{f}} h = [\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \ldots, \frac{\partial h}{\partial x_n}]\mathbf{f}$$

$$ad_{\mathbf{f}}\, \mathbf{g} = [\mathbf{f}, \mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g}$$

$$ad_{\mathbf{f}^0}\, \mathbf{g} = \mathbf{g}, \qquad ad_{\mathbf{f}^{k+1}}\, \mathbf{g} = [\mathbf{f}, ad_{\mathbf{f}^k}\, \mathbf{g}], \;\; k = 0, 1, \ldots$$

    (b) Relative Degree
        For the single input/single output affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \qquad y = h(\mathbf{x})$$

        the relative degree (r) is the first time-derivative of y for which u appears in the expression for the derivative. For this to happen the following must be satisfied

$$L_{\mathbf{g}} L_{\mathbf{f}^{k-1}} h(\mathbf{x}) = 0 \qquad k = 1, 2, \ldots, r-1$$

$$L_{\mathbf{g}} L_{\mathbf{f}^{r-1}} h(\mathbf{x}_e) \neq 0$$

(c) Normal Form
Associated with an output of relative degree r, there exists a transformation to normal form

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= A + Bu \\
\dot{\eta}_{r+1} &= q_{r+1}(\mathbf{z}) \\
\dot{\eta}_{r+2} &= q_{r+2}(\mathbf{z}) \\
&\vdots \\
\dot{\eta}_n &= q_n(\mathbf{z}) \\
y &= \xi_1
\end{aligned}
$$

where

$$
\boldsymbol{\Phi}(\mathbf{x}) = \mathbf{z} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}
$$

with the linear part $\xi_i = L_{\mathbf{f}^{i-1}} h(\mathbf{x}),\ i = 1, 2, \ldots r$ and the remnant η-dynamics chosen not to depend on u.

(d) Exact Feedback Linearisation
For a single input affine control system, the linearising transformation and linearising control are

$$
\mathbf{z} = \Phi(\mathbf{x}), \qquad u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v
$$

where the latter is obtained by solving $A(\mathbf{x}) + B(\mathbf{x})u = v$.
Algorithm:

i. compute the set of vector fields

$$
F_n = \{\mathbf{g},\ ad_{\mathbf{f}}\,\mathbf{g}, ad_{\mathbf{f}^2}\,\mathbf{g}, \ldots, ad_{\mathbf{f}^{n-1}}\,\mathbf{g}\}
$$

ii. check that there exists a neighbourhood $\mathcal{N}$ of $\mathbf{0}$ such that (i) F_n is linearly independent (equivalent to the system being controllable) and (ii) F_{n-1} is involutive (meaning that a certain set of pdes can be solved). If so then...

iii. Solve the system of pdes

$$
\begin{aligned}
L_{ad_{\mathbf{f}^i}\,\mathbf{g}}\, z_1 &= 0, \qquad\qquad i = 0, 1, \ldots, n-2 \\
L_{ad_{\mathbf{f}^{n-1}}\,\mathbf{g}}\, z_1 &\neq 0
\end{aligned}
$$

for z_1 the first component of $\mathbf{z}$.

iv. Compute the state transformation

$$
\mathbf{z}(\mathbf{x}) = \Phi(\mathbf{x}) = \begin{pmatrix} z_1, & L_{\mathbf{f}}\, z_1, & L_{\mathbf{f}^2}\, z_1, & \ldots, & L_{\mathbf{f}^{n-1}}\, z_1 \end{pmatrix}^T
$$

and the functions

$$
\beta(\mathbf{x}) = \frac{1}{B(\mathbf{x})}, \qquad\qquad \alpha(\mathbf{x}) = -\frac{A(\mathbf{x})}{B(\mathbf{x})}.
$$

where $B(\mathbf{x}) = L_{\mathbf{g}} L_{\mathbf{f}^{n-1}}\, z_1 = L_{\mathbf{g}} z_n$ and $A(\mathbf{x}) = L_{\mathbf{f}^n}\, z_1 = L_{\mathbf{f}} z_n$