

## 1. Linear State Equations

The LTI continuous-time equation  $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ ,  $\mathbf{x}(t_0) = \mathbf{x}_0$  has solution

$$\mathbf{x}(t) = e^{A(t-t_0)}\{\mathbf{x}_0 + \int_{t_0}^t e^{A(t_0-\tau)}B\mathbf{u}(\tau) d\tau\}$$

2. Computing  $A^k$  and  $e^{At}$  when  $A$  is diagonalisable (  $A = E\Lambda E^{-1}$  )

$$A^k = E\Lambda^k E^{-1}$$

$$e^{At} = E \left( \sum_{j=0}^{\infty} \frac{\Lambda^j t^j}{j!} \right) E^{-1} = E e^{\Lambda t} E^{-1}$$

3. *Lyapunov's* Direct Method for Linear systems

The (continuous-time) *Lyapunov* Equation is

$$A^T P + P A = -Q$$

## 4. Transformations

The transformation  $\mathbf{x} = T\mathbf{z}$  transforms  $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ ,  $\mathbf{y} = C\mathbf{x}$  to  $\dot{\mathbf{z}} = T^{-1}AT\mathbf{z} + T^{-1}B\mathbf{u}$ ,  $\mathbf{y} = C T\mathbf{x}$ .

## 5. Controllability, Observability, &amp; Duality

(a) The Controllability  $U$  and Observability  $V$  matrices are given by:

$$U = (B, AB, A^2B, \dots, A^{n-1}B) \quad V = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

(b) *Popov-Belevitch-Hautus* (PBH) test for controllability:

$$M \triangleq [\lambda I - A, B]$$

is of rank  $n$  for all  $\lambda$  if and only if  $(A, B)$  is completely controllable.

(c) The PBH test for observability:

$$N \triangleq \begin{bmatrix} C \\ \lambda I - A \end{bmatrix}$$

is of rank  $n$  for all  $\lambda$  if and only if  $(A, C)$  is completely observable.

- (d) Controllable Canonical Form for a Single Input System with characteristic polynomial  $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$ :

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

- (e) The system  $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ ,  $\mathbf{y} = C\mathbf{x}$  has dual  $\dot{\mathbf{x}}^* = A^T\mathbf{x}^* + C^T\mathbf{u}^*$ ,  $\mathbf{y}^* = B^T\mathbf{x}^*$

## 6. Pole Placement for Single Input Systems

- (a) Transformation to ccf approach.

The transformation  $\hat{T} = U\hat{U}^{-1}$ , where  $U$  and  $\hat{U}$  are the controllability matrices of the original and transformed systems respectively, transforms a completely controllable system into controllable canonical form.

The matrix  $K = \hat{K}\hat{T}^{-1}$  where  $\hat{k}_i = a_{i-1} - \hat{a}_{i-1}$ ,  $i = 1, 2, \dots, n$  is the required feedback matrix.

- (b) *Ackermann's* Formula

The formula is

$$K = -[0, 0, \dots, 0, 1]U^{-1}\chi_{cl}^{des}(A)$$

where  $U = [b, Ab, \dots, A^{n-1}b]$  is the controllability matrix of the open-loop system, and  $\chi_{cl}^{des}(A) = A^n + \hat{a}_{n-1}A^{n-1} + \dots + \hat{a}_1A + \hat{a}_0I$  is the desired characteristic polynomial of the closed-loop system matrix evaluated at the open-loop system matrix.

## 7. Luenberger Observers

The continuous-time full order *Luenberger* observer is

$$\dot{\mathbf{z}} = (A + LC)\mathbf{z} + B\mathbf{u} - L\mathbf{y}$$

## 8. Decoupling

Let  $d_i$  be the smallest nonnegative integer such that  $c_iA^{d_i}B \neq 0$ . Then

$$\hat{\mathbf{y}} = \begin{pmatrix} c_1A^{d_1+1} \\ c_2A^{d_2+1} \\ \vdots \\ c_mA^{d_m+1} \end{pmatrix} \mathbf{x} + \begin{pmatrix} c_1A^{d_1}B \\ c_2A^{d_2}B \\ \vdots \\ c_mA^{d_m}B \end{pmatrix} \mathbf{u}$$

where  $\hat{y}_i = \frac{d^{d_i+1}y_i}{dt^{d_i+1}}$ . The *remnant*, if it exists, is of order  $q = n - \sum (d_i + 1)$ .

## 9. Characteristic Polynomials and Stability

- (a) The 2nd degree polynomial

$$\lambda^2 + a_1\lambda + a_0$$

has all its roots (eigenvalues, poles) in the left half plane (LHP) iff

$$a_1 > 0, \quad a_0 > 0.$$

(b) The 3rd degree polynomial

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

has all its roots (eigenvalues, poles) in the LHP iff

$$a_2 > 0, \quad a_1 > 0, \quad a_0 > 0 \quad \text{and} \quad a_2a_1 > a_0.$$

10. *Hamilton-Jacobi-Bellman* (HJB) Equation for infinite horizon problems.

The stabilising control that minimises  $J = \int_{t_0}^{\infty} L(\mathbf{x}, \mathbf{u}) dt$  for the system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ ,  $\mathbf{x}(t_0) = \mathbf{x}_0$  is given by the solution of

$$0 = \min_{\mathbf{u}(t)} \left\{ L(\mathbf{x}, \mathbf{u}) + \left( \frac{\partial V}{\partial \mathbf{x}} \right)^T \mathbf{f}(\mathbf{x}, \mathbf{u}) \right\}$$

where  $V(\mathbf{x}) = \min_{\mathbf{u}(\cdot)} J$ .

11. Linear Quadratic (LQ) control

The minimum value of

$$J(\mathbf{u}) = \mathbf{x}^T(t_f) S \mathbf{x}(t_f) + \int_{t_0}^{t_f} \mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u} dt$$

for the system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by  $V(\mathbf{x}_0, t_0)$  where  $V(\mathbf{x}, t) = \mathbf{x}^T P \mathbf{x}$  and  $P = P^T$  is the solution of the matrix *Riccati* equation (MRE)

$$-\dot{P} = Q + A^T P + P A - P B R^{-1} B^T P \quad P(t_f) = S$$

Furthermore  $\mathbf{u}(t) = -R^{-1} B^T P \mathbf{x}(t)$ .

Additionally, if  $t_f \rightarrow \infty$ , then for the LTI system with  $(A, B)$  completely controllable and  $(A, C)$  completely observable where  $C^T C = Q$ ,  $P$  is the positive definite solution of the continuous-time algebraic *Riccati* equation (CARE)

$$0 = Q + A^T P + P A - P B R^{-1} B^T P$$

12. A modified *LaSalle* Invariance Principle: For the system

$$\dot{x} = \mathbf{f}(\mathbf{x}), \quad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

if (i)  $V$  is positive definite, (ii)  $\frac{dV}{dt}$  is negative semi-definite and (iii) the union of all invariant sets with  $\frac{dV}{dt} = 0$  is  $M = \{\mathbf{0}\}$ , then  $\mathbf{0}$  is asymptotically stable. In particular for the system

$$\ddot{x} + b(\dot{x}) + c(x) = 0,$$

where  $b$  and  $c$  are continuous functions with the same sign as their arguments, the origin  $\mathbf{0}$  is asymptotically stable. This may be shown using

$$V(\mathbf{x}) = \frac{1}{2} x_2^2 + \int_0^{x_1} c(s) ds$$

### 13. Control *Lyapunov* Functions

$V(\mathbf{x})$  is a control *Lyapunov* function for the affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \quad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

if (i) it is positive definite and (ii)

$$b(\mathbf{x}) = \mathbf{0} \quad \Rightarrow \quad a(\mathbf{x}) < 0, \quad \mathbf{x} \neq \mathbf{0}$$

where

$$\frac{dV}{dt} = \left( \frac{\partial V}{\partial \mathbf{x}} \right)^T \dot{\mathbf{x}} = \underbrace{\left( \frac{\partial V}{\partial \mathbf{x}} \right)^T \mathbf{f}(\mathbf{x})}_{a(\mathbf{x})} + \underbrace{\left( \frac{\partial V}{\partial \mathbf{x}} \right)^T \mathbf{g}(\mathbf{x}) \mathbf{u}}_{b(\mathbf{x})}$$

*Sontag's* formula: for a single input system a stabilising control is

$$u = k(\mathbf{x}) \triangleq \begin{cases} -\frac{a(\mathbf{x}) + \sqrt{a^2(\mathbf{x}) + b^4(\mathbf{x})}}{b(\mathbf{x})}, & \text{if } b(\mathbf{x}) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

### 14. Geometric Control

(a) Differential Geometry notations:

$$L_{\mathbf{f}}h = \left[ \frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \dots, \frac{\partial h}{\partial x_n} \right] \mathbf{f}$$

$$ad_{\mathbf{f}} \mathbf{g} = [\mathbf{f}, \mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \mathbf{g}$$

$$ad_{\mathbf{f}^0} \mathbf{g} = \mathbf{g}, \quad ad_{\mathbf{f}^{k+1}} \mathbf{g} = [\mathbf{f}, ad_{\mathbf{f}^k} \mathbf{g}], \quad k = 0, 1, \dots$$

(b) Relative Degree

For the single input/single output affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \quad y = h(\mathbf{x})$$

the relative degree ( $r$ ) is the first time-derivative of  $y$  for which  $u$  appears in the expression for the derivative. For this to happen the following must be satisfied

$$L_{\mathbf{g}}L_{\mathbf{f}^{k-1}}h(\mathbf{x}) = 0 \quad k = 1, 2, \dots, r-1$$

$$L_{\mathbf{g}}L_{\mathbf{f}^{r-1}}h(\mathbf{x}_e) \neq 0$$

(c) Normal Form

Associated with an output of relative degree  $r$ , there exists a transformation to normal form

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= A + Bu \\ \dot{\eta}_{r+1} &= q_{r+1}(\mathbf{z}) \\ \dot{\eta}_{r+2} &= q_{r+2}(\mathbf{z}) \\ &\vdots \\ \dot{\eta}_n &= q_n(\mathbf{z}) \\ y &= \xi_1 \end{aligned}$$

where

$$\Phi(\mathbf{x}) = \mathbf{z} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

with the linear part  $\xi_i = L_{\mathbf{f}^{i-1}} h(\mathbf{x})$ ,  $i = 1, 2, \dots, r$  and the remnant  $\eta$ -dynamics chosen not to depend on  $u$ .

(d) Exact Feedback Linearisation

For a single input affine control system, the linearising transformation and linearising control are

$$\mathbf{z} = \Phi(\mathbf{x}), \quad u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v$$

where the latter is obtained by solving  $A(\mathbf{x}) + B(\mathbf{x})u = v$ .

Algorithm:

- i. compute the set of vector fields

$$F_n = \{\mathbf{g}, ad_{\mathbf{f}} \mathbf{g}, ad_{\mathbf{f}^2} \mathbf{g}, \dots, ad_{\mathbf{f}^{n-1}} \mathbf{g}\}$$

- ii. check that there exists a neighbourhood  $\mathcal{N}$  of  $\mathbf{0}$  such that (i)  $F_n$  is linearly independent (equivalent to the system being controllable) and (ii)  $F_{n-1}$  is involutive (meaning that a certain set of pdes can be solved). If so then...

- iii. Solve the system of pdes

$$\begin{aligned} L_{ad_{\mathbf{f}^i} \mathbf{g}} z_1 &= 0, & i &= 0, 1, \dots, n-2 \\ L_{ad_{\mathbf{f}^{n-1}} \mathbf{g}} z_1 &\neq 0 \end{aligned}$$

for  $z_1$  the first component of  $\mathbf{z}$ .

- iv. Compute the state transformation

$$\mathbf{z}(\mathbf{x}) = \Phi(\mathbf{x}) = (z_1, L_{\mathbf{f}} z_1, L_{\mathbf{f}^2} z_1, \dots, L_{\mathbf{f}^{n-1}} z_1)^T$$

and the functions

$$\beta(\mathbf{x}) = \frac{1}{B(\mathbf{x})}, \quad \alpha(\mathbf{x}) = -\frac{A(\mathbf{x})}{B(\mathbf{x})}.$$

where  $B(\mathbf{x}) = L_{\mathbf{g}} L_{\mathbf{f}^{n-1}} z_1 = L_{\mathbf{g}} z_n$  and  $A(\mathbf{x}) = L_{\mathbf{f}^n} z_1 = L_{\mathbf{f}} z_n$