State Estimators

Luenberger was the first to realise that practically any other LTI system (which we'll call the "observer") could act as a estimator of the state of a completely observable (CO) LTI target system (herein called the "system").

1 Full order observers for CO LTI systems

1.1 Continuous-time systems

For the LTI system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{1}$$

$$\mathbf{y} = C\mathbf{x} \tag{2}$$

we hook up the observer as shown in Fig. 1 [It is a standard LTI system with state vector $\mathbf{z}$ and input vector $\begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}$].

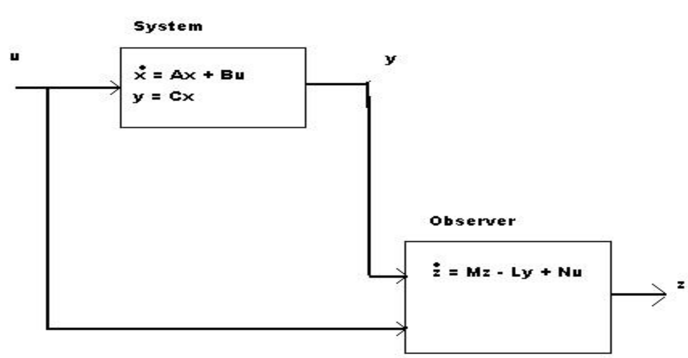

Figure 1: System - Observer Configuration

$$\begin{aligned} \dot{\mathbf{z}} &= M\mathbf{z} + \begin{pmatrix} -L & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix} \\ &= M\mathbf{z} - L\mathbf{y} + N\mathbf{u} \end{aligned} \tag{3}$$

Our intent is to choose the matrices M, L and N such that $\mathbf{z}(t)$ is a "estimate" of $\mathbf{x}(t)$: in particular, we'll design the observer so that $\mathbf{z}(t) \to \mathbf{x}(t)$ as $t \to \infty$.

Define the error

$$\begin{aligned}
\mathbf{e} &\triangleq \mathbf{z}-\mathbf{x} \\
\Rightarrow \dot{\mathbf{e}} &= \dot{\mathbf{z}}-\dot{\mathbf{x}} \\
&= M\mathbf{z}-L\mathbf{y}+N\mathbf{u}-A\mathbf{x}-B\mathbf{u} \\
&= M\mathbf{z}-LC\mathbf{x}+N\mathbf{u}-A\mathbf{x}-B\mathbf{u}, \qquad \text{Choose} \quad N=B \\
&= M\mathbf{z}-(A+LC)\mathbf{x}, \qquad \text{Choose} \quad M=A+LC \\
&= (A+LC)\mathbf{e}
\end{aligned}$$

Since $(A,\ C)$ is CO, we know by Duality that there exists a matrix L such that $A+LC$ can have any set of desired eigenvalues.[1] In particular, we can choose these eigenvalues so that they all have negative real parts, and thus $\mathbf{e}(t) \to \mathbf{0}$ as $t \to \infty$. In practice, the eigenvalues of $A+LC$ are chosen to be faster than those of the original system. With these choices, the description of the observer becomes

$$\dot{\mathbf{z}} = (A+LC)\mathbf{z} - L\mathbf{y} + B\mathbf{u} \tag{4}$$

The observer designed above is "full order": the observer has the same order as the system. It is possible to design reduced order observers which take account of the fact that $\mathbf{y}$ itself measures some of the components of $\mathbf{x}$ (or functions thereof) - thus a reduced order observer would only estimate the missing components of the state.

1.2 Discrete-time Systems

The design of the observer follows along the same lines as that for the continuous-time case.

The system is

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k \tag{5}$$

$$\mathbf{y}_k = C\mathbf{x}_k \tag{6}$$

The observer is described by

$$\mathbf{z}_{k+1} = (A+LC)\mathbf{z}_k - L\mathbf{y}_k + B\mathbf{u}_k \tag{7}$$

while the error equation is

$$\mathbf{e}_{k+1} = (A+LC)\mathbf{e}_k$$

Of course, choice of the locations for the observer poles might require extra care.