State Transformations

A state transformation or "a change of variables" is used to simplify a calculation and/or to gain insight into a phenomenon. Consider the situation of changing from the existing state representation $\mathbf{x}$ to a new state representation $\mathbf{z}$, both $n \times 1$ vectors. The transformation must be invertible and can be represented by

$$\mathbf{x} = \mathbf{T}(\mathbf{z}) \quad \Leftrightarrow \quad \mathbf{z} = \mathbf{T}^{-1}(\mathbf{x})$$

In the case of a linear state transformation, $\mathbf{T}$ is a $n \times n$ invertible matrix. For nonlinear transformations, we usually require that $\mathbf{T}$ be a homeomorphism (both $\mathbf{T}$ and $\mathbf{T}^{-1}$ are continuous).

Denoting the *Jacobian* matrix of $\mathbf{T}$ by $D\mathbf{T}$ we get for a flow that this implies

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \Rightarrow \frac{d\,\mathbf{T}(\mathbf{z})}{dt} &= \mathbf{f}(\mathbf{T}(\mathbf{z})) \\ \Rightarrow D\mathbf{T}\dot{\mathbf{z}} &= \mathbf{f}(\mathbf{T}(\mathbf{z})) \\ \Rightarrow \dot{\mathbf{z}} &= \underbrace{(D\mathbf{T})^{-1}(\mathbf{f}(\mathbf{T}(\mathbf{z})))}_{\mathbf{g}(\mathbf{z})} \\ \Rightarrow \dot{\mathbf{z}} &= \mathbf{g}(\mathbf{z}) \end{aligned}$$

where $\mathbf{g} = (D\mathbf{T})^{-1}\mathbf{f} \circ \mathbf{T}$.

Similarly for a map

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

becomes

$$\mathbf{z}' = \mathbf{g}(\mathbf{z})$$

where it can be shown that this time $\mathbf{g} = \mathbf{T}^{-1} \circ \mathbf{f} \circ \mathbf{T}$. In this case $\mathbf{f}$ and $\mathbf{g}$ are said to be topologically conjugate maps, and dynamical features of one are preserved by the other under conjugation, i.e. fixed points of one are mapped into the other, periodic orbits of one are mapped into the other, etc.

1 Linear State Transformations & Linear Systems

Most of the analysis holds for flows and maps. We'll illustrate the ideas with flows unless otherwise stated.

The linear state transformation $\mathbf{x} = T\mathbf{z}$ converts the linear flow

$$\dot{\mathbf{x}} = A\mathbf{x}$$

to

$$\begin{aligned} T\dot{\mathbf{z}} &= AT\mathbf{z} \\ \Rightarrow \quad \dot{\mathbf{z}} &= \underbrace{T^{-1}AT}_{\hat{A}}\mathbf{z} \qquad (1) \\ \Rightarrow \quad \dot{\mathbf{z}} &= \hat{A}\mathbf{z} \end{aligned}$$

Recall that the matrices A and $\hat{A}$ are similar and hence have the same spectrum. Thus this type of transformation leaves the eigenvalues of the state matrix unchanged.

1.1 The Modal Transformation

If A is diagonalisable (i.e. similar to a diagonal matrix), then there exists an invertible matrix E such that

$$A = E\Lambda E^{-1} \qquad \Leftrightarrow \qquad E^{-1}AE = \Lambda \tag{2}$$

where Λ is a diagonal matrix whose diagonal entries are the eigenvalues of A is some order, and E is a matrix whose columns are the corresponding eigenvectors in the same order.

Then the modal transformation $\mathbf{x} = E\mathbf{z}$ converts the system

$$\dot{\mathbf{x}} = A\mathbf{x}$$

to

$$\dot{\mathbf{z}} = \Lambda\mathbf{z}$$

or

$$\dot{z}_i = \lambda_i z_i \qquad i = 1, 2, \cdots, n \tag{3}$$

In this representation, each component of the transformed state vector $\mathbf{z}$ is decoupled from the other components, thus making computations more transparent.

If Eq (3) is solved with initial condition $\mathbf{z}(0) = [z_1(0), z_2(0), \ldots z_n(0)]$ then the solution is

$$z_i(t) = e^{\lambda_i t} z_i(0), \qquad i = 1, 2, \cdots, n \tag{4}$$

1.1.1 Complex Eigenvalues

Eq (4) has a natural interpretation when all the eigenvalues are real. What happens if there are complex eigenvalues? These occur in conjugate pairs. Let's consider the 2-d case where $\lambda = \sigma \pm \omega i$ with corresponding eigenvectors $\mathbf{e}_\lambda = \mathbf{e}_R \pm \mathbf{e}_I i$. It can be shown that the matrix $T_c = [\mathbf{e}_R, \mathbf{e}_I]$ is invertible and that

$$\Omega \triangleq \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} = T_c^{-1} A T_c \tag{5}$$

Applying the transformation $\mathbf{x} = T_c\mathbf{z}$ to the system $\dot{\mathbf{x}} = A\mathbf{x}$ gives

$$\dot{\mathbf{z}} = \Omega\mathbf{z} \tag{6}$$

which has solution (for initial state $\mathbf{z}(0)$)

$$\mathbf{z}(t) = e^{\sigma t} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix} \mathbf{z}(0)$$

Alternatively, transforming Eq (6) using polar coordinates gives

$$\begin{aligned} \dot{r} &= \sigma r \qquad \Rightarrow \qquad r(t) = r_0 e^{\sigma t} \\ \dot{\theta} &= -\omega \qquad \Rightarrow \qquad \theta(t) = \theta_0 - \omega t \end{aligned}$$

i.e. the solution trajectory spirals back to the origin if $\sigma < 0$, spirals away from the origin if $\sigma > 0$, and circles the origin if $\sigma = 0$ (this last conclusion holds only if A is associated with a truly linear system and not with a linearised *Jacobian*).