Stability

For the autonomous system described by the differential equation

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{f}(\mathbf{x}(t)), \qquad 0 \leq t < \infty \tag{1}$$

let $\mathbf{x}(t) = \mathbf{\Phi}(t, \mathbf{x}_0)$ represent its solution starting in the initial state $\mathbf{x}(0) = \mathbf{x}_0$. Such a solution curve is called a *trajectory* or *orbit*. We assume that $\mathbf{f}$ is such that $\mathbf{\Phi}(t, \mathbf{x}_0)$ exists and is unique, and we note that $\mathbf{\Phi}(0, \mathbf{x}_0) = \mathbf{x}_0$.

A solution of Equation (1) is said to be **stable** in the sense of *Lyapunov* if other solutions which start close to $\mathbf{x}_0$, at $\tilde{\mathbf{x}}_0$ say, stay close to $\mathbf{x}(t)$ for all time. More formally, for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\parallel \mathbf{\Phi}(0, \tilde{\mathbf{x}}_0) - \mathbf{\Phi}(0, \mathbf{x}_0) \parallel < \delta \Rightarrow \parallel \mathbf{\Phi}(t, \tilde{\mathbf{x}}_0) - \mathbf{\Phi}(t, \mathbf{x}_0) \parallel < \epsilon \tag{2}$$

for all $t \geq 0$. We note that this definition does not require that $\mathbf{\Phi}(t, \tilde{\mathbf{x}}_0)$ converge to $\mathbf{\Phi}(t, \mathbf{x}_0)$.

A solution of Equation (1) is said to be **asymptotically stable** if

(1) it is stable, and

(2) $\lim_{t\to\infty} \mathbf{\Phi}(t, \tilde{\mathbf{x}}_0) = \lim_{t\to\infty} \mathbf{\Phi}(t, \mathbf{x}_0)$

Finally, a solution of Equation(1) is said to be **unstable** if it is not stable.

All these definitions are for local phenomena: for $\tilde{\mathbf{x}}_0$ "close" to $\mathbf{x}_0$. If the definitions hold for all $\tilde{\mathbf{x}}_0$, we talk about global stability, etc.

Of particular interest for stability analysis are regular trajectories like

- equilibria: $\mathbf{x}(t) = \mathbf{\Phi}(t, \mathbf{x}_e) \equiv \mathbf{x}_e$ given by solutions of $\mathbf{f}(\mathbf{x}_e) = \mathbf{0}$, or
- periodic orbits $\mathbf{x}(t + T) = \mathbf{x}(t)$, for fixed $T > 0$.

Indeed, since it arises so frequently in the study of Dynamical Systems, let us restate the formal definition for stability of the equilibrium state (or fixed point) $\mathbf{x}_e$ as:

$\mathbf{x}_e$ is said to be **stable** in the sense of *Lyapunov* if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\parallel \mathbf{\Phi}(0, \tilde{\mathbf{x}}_0) - \mathbf{x}_e \parallel < \delta \Rightarrow \parallel \mathbf{\Phi}(t, \tilde{\mathbf{x}}_0) - \mathbf{x}_e \parallel < \epsilon \tag{3}$$

for all $t \geq 0$.

Similar definitions for stability, asymptotic stability and instability hold for the autonomous system described by the difference equation

$$\mathbf{x}(k+1) = \mathbf{F}(\mathbf{x}(k)), \qquad k = 0, 1, 2, \ldots \tag{4}$$

Let $\mathbf{x}(k) = \mathbf{\Phi}(k, \mathbf{x}_0)$ represent its solution/trajectory/orbit starting from $\mathbf{x}(0) = \mathbf{x}_0$. We note that $\mathbf{\Phi}(k, \mathbf{x}_0) = \mathbf{F}^k(\mathbf{x}_0)$ is the k-fold composition of $\mathbf{F}$ with itself acting on $\mathbf{x}_0$.

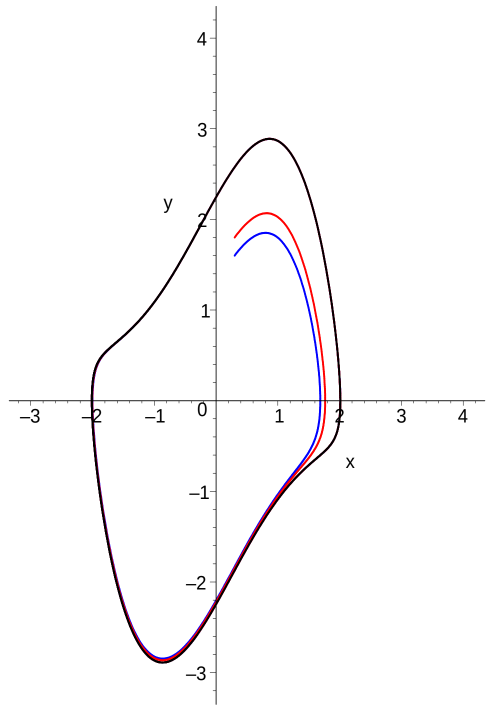

Figure 1: Two initially close trajectories (red and blue) converge to the same limit cycle (black)

Stability Criteria for fixed points of Linear Systems

For linear systems, local stability is synonymous with global stability, and because of their structure we can tease out some implications of the above definitions.

The Linear Map $\mathbf{x}(k+1) = A\mathbf{x}(k)$ has fixed point $\mathbf{x}_e = \mathbf{0}$, and for initial state $\mathbf{x}_0$ has solution $\mathbf{x}(k) = A^k\mathbf{x}_0$. If we assume that A is diagonalisable, we can write this as

$$\begin{aligned} \mathbf{x}(k) &= E\Lambda^k E^{-1}\mathbf{x}_0 \\ \text{or} \qquad x_i(k) &= \sum_{j=1}^{n} c_{i,j}\lambda_j^k \qquad\qquad i = 1, 2, \ldots, n \end{aligned} \tag{5}$$

where the $c_{i,j}$ terms depend on the entries in E and $\mathbf{x}_0$, i.e. they can take any real value as $\mathbf{x}_0$ varies. If A is not diagonalisable, then the $c_{i,j}$ which are constants in Equation (5) become polynomials in k of degree at most $n-1$. In either case, the limiting behaviour as $k \to \infty$ is determined by the λ^k terms.

By inspection of Equation (5) we see

(1) a necessary and sufficient condition for $\lim_{k\to\infty} \mathbf{x}(k) = \mathbf{x}_e = \mathbf{0}$ is that

$$|\lambda_i| < 1, \qquad i = 1, 2, \ldots, n$$

(2) a sufficient condition for some trajectory to leave the neighbourhood of $\mathbf{x}_e = \mathbf{0}$ is that there exist λ_j with $|\lambda_j| > 1$

(3) if $|\lambda_i| \leq 1$ for $i = 1, 2, \ldots, n$ with some $|\lambda_j| = 1$ then whether $\mathbf{x}(k)$ stays close to $\mathbf{0}$ or diverges with increasing time depends on the structure of the relevant $c_{i,j}$ terms.

On the other hand, the linear flow $\dot{\mathbf{x}} = A\mathbf{x}$ also has equilibrium state $\mathbf{x}_e = \mathbf{0}$, and for initial state $\mathbf{x}_0$ has solution $\mathbf{x}(t) = e^{At}\mathbf{x}_0$. If we again assume that A is diagonalisable, we can write this as

$$\begin{aligned} \mathbf{x}(t) &= E e^{\Lambda t} E^{-1} \mathbf{x}_0 \\ \text{or} \quad x_i(t) &= \sum_{j=1}^{n} c_{i,j} e^{\lambda_j t} \qquad i = 1, 2, \ldots, n \end{aligned} \tag{6}$$

where again the $c_{i,j}$ terms depend on the entries in E and $\mathbf{x}_0$. If A is not diagonalisable, then the $c_{i,j}$ which are constants in Equation (6) become polynomials in t of degree at most $n-1$. In either case, the limiting behaviour as $t \to \infty$ is determined by the $e^{\lambda t}$ terms.

By inspection of Equation (6) we see

(1) a necessary and sufficient condition for $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_e = \mathbf{0}$ is that

$$\begin{aligned} e^{\lambda_i t} &< 1, \qquad t \to \infty, \qquad i = 1, 2, \ldots, n \\ \Rightarrow \Re(\lambda_i) &< 0, \qquad i = 1, 2, \ldots, n \end{aligned}$$

(2) a sufficient condition for some trajectory to leave the neighbourhood of $\mathbf{x}_e = \mathbf{0}$ is that there exist λ_j with $\Re(\lambda_j) > 0$

(3) if $\Re(\lambda_i) \leq 0$ for $i = 1, 2, \ldots, n$ with some $\Re(\lambda_j) = 0$ then whether $\mathbf{x}(t)$ stays close to $\mathbf{0}$ or diverges with increasing time depends on the structure of the relevant $c_{i,j}$ terms.