State Transformations

A state transformation or "a change of variables" is used to simplify a calculation and/or to gain insight into a phenomenon. Consider the situation of changing from the existing state representation $\mathbf{x}$ to a new state representation $\mathbf{z}$, both $n \times 1$ vectors. The transformation must be invertible and can be represented by

$$\mathbf{x} = \mathbf{T}(\mathbf{z}) \quad \Leftrightarrow \quad \mathbf{z} = \mathbf{T}^{-1}(\mathbf{x})$$

In the case of a linear state transformation, $\mathbf{T}$ is a $n \times n$ invertible matrix. For nonlinear transformations, we will use the notation $\mathbf{\Phi} = \mathbf{T}^{-1}$, and will only deal with homeomorphisms, i.e. $\mathbf{\Phi}$ and $\mathbf{\Phi}^{-1}$ are both continuous.

1 Linear State Transformations & Linear Systems

The linear state transformation $\mathbf{x} = T\mathbf{z}$ converts the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x}$$

to

$$\begin{aligned} T\dot{\mathbf{z}} &= AT\mathbf{z} + B\mathbf{u}, \qquad \mathbf{y} = CT\mathbf{z} \\ \Rightarrow \quad \dot{\mathbf{z}} &= \underbrace{T^{-1}AT}_{\hat{A}}\mathbf{z} + \underbrace{T^{-1}B}_{\hat{B}}\mathbf{u}, \quad \mathbf{y} = \underbrace{CT}_{\hat{C}}\mathbf{z} \\ \Rightarrow \quad \dot{\mathbf{z}} &= \hat{A}\mathbf{z} + \hat{B}\mathbf{u}, \qquad \mathbf{y} = \hat{C}\mathbf{z} \end{aligned}$$

Recall that the matrices A and $\hat{A}$ are similar and hence have the same spectrum. Thus this type of transformation leaves the eigenvalues of the state matrix unchanged.

1.1 The Modal Transformation

If A is diagonalisable (i.e. similar to a diagonal matrix), then there exists an invertible matrix E such that

$$A = E\Lambda E^{-1} \quad \Leftrightarrow \quad E^{-1}AE = \Lambda$$

where Λ is a diagonal matrix whose diagonal entries are the eigenvalues of A is some order, and E is a matrix whose columns are the corresponding eigenvectors in the same order.

Then the modal transformation $\mathbf{x} = E\mathbf{z}$ converts the system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x}$$

to

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \hat{B}\mathbf{u}, \qquad \mathbf{y} = \hat{C}\mathbf{z}$$

or

$$\dot{z}_i = \lambda_i z_i + \hat{B}_i\mathbf{u}, \qquad \mathbf{y} = \hat{C}_1 z_1 + \hat{C}_2 z_2 + \cdots + \hat{C}_n z_n, \qquad i = 1, 2, \cdots, n$$

where $\hat{B} = E^{-1}B$, $\hat{B}_i$ is i-th row of $\hat{B}$, $\hat{C} = CE$ and $\hat{C}_i$ is i-th column of $\hat{C}$. In this representation, each component of the transformed state vector $\mathbf{z}$ is decoupled from the other components, thus making computations more transparent.

1.2 Transformation to Controllable Canonical Form (CCF)

The $n \times nr$ controllability matrix U is defined by

$$U \triangleq \left[B, AB, A^2B, \ldots, A^{n-1}B\right]$$

It is well known that if U is of full rank, then the linear system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ is completely controllable (CC).

For a single input ($r = 1$) CC system, U is a square matrix and so being of full rank is equivalent to U^{-1} existing. For such a system there exists a state transformation $\mathbf{x} = T_c\mathbf{z}$ which transforms the system to CCF:

$$\dot{\mathbf{z}} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}}_{A_c} \mathbf{z} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{b_c} u$$

or

$$\dot{\mathbf{z}} = A_c\mathbf{z} + b_c u$$

where the characteristic polynomial of A is $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$.

A_c is said to be a *companion* form or *Frobenius* form matrix.

The transformation T_c is given by $T_c = UU_c^{-1}$ where U_c is the controllability matrix of the CCF system, i.e.

$$U_c = \left[b_c, A_cb_c, A_c^2b_c, \ldots, A_c^{n-1}b_c\right]$$

Exercise: Verify that U_c is invertible.