Pole Placement

1 Linear State Feedback

The ideas in this chapter are presented for continuous-time LTI systems. They equally well hold for discrete-time LTI systems.
For a system with r inputs, the control

$$\begin{aligned} \mathbf{u} &= K\mathbf{x} + \mathbf{v} \\ \text{or } u_i &= k_{i,1}x_1 + k_{i,2}x_2 + \cdots + k_{i,n}x_n + v_i, \qquad i = 1, 2, \ldots, r \end{aligned}$$

is generated by linear feedback of the state, where $\mathbf{v}$ is a new reference input. It changes the Open-Loop system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

to Closed-Loop form

$$\dot{\mathbf{x}} = (A + BK)\mathbf{x} + B\mathbf{v}$$

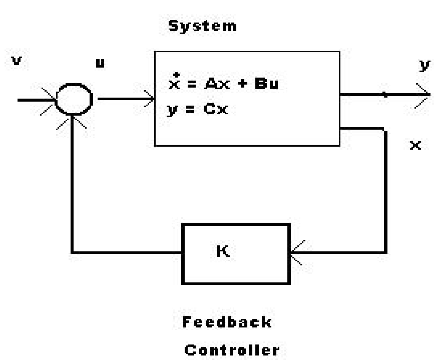

Figure 1: Linear State Feedback

The question immediately arises: what effect does choice of K have on the behaviour of the closed-loop system?

One of the fundamental results of state space control theory is: If the pair (A, B) is completely controllable (CC), then the closed-loop poles or eigenvalues of $A + BK$ can be arbitrarily placed in the complex plane [1] by linear state feedback, i.e. by appropriately choosing K. This is referred to as the Pole Placement or Eigenvalue Assignment problem.
We will establish this result by constructive means in the context of single input sytems.

2 Pole Placement for Single Input systems

Since we are assuming that the pair (A, b) is CC, we know that there exists a state transformation $\mathbf{x} = \hat{T}\hat{\mathbf{x}}$ which transforms the system to controllable canonical form (- see the chapter on controllability), i.e.

$$\dot{\mathbf{x}} = A\mathbf{x} + bu \qquad \overset{\mathbf{x}=\hat{T}\hat{\mathbf{x}}}{\rightarrow} \qquad \dot{\hat{\mathbf{x}}} = \hat{A}\hat{\mathbf{x}} + \hat{b}u \tag{1}$$

where

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \qquad \hat{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

and

$$\chi_{ol}(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{3}$$

is the characteristic polynomial of A and open-loop characteristic polynomial.
Let's assume that we wish to place the closed loop poles at $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$; then the desired closed loop characteristic polynomial is

$$\begin{aligned} \chi_{cl}^{des}(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \\ &= \lambda^n + \hat{a}_{n-1}\lambda^{n-1} + \cdots + \hat{a}_1\lambda + \hat{a}_0 \end{aligned} \tag{4}$$

The feedback control

$$u = K\mathbf{x} + v = k_1x_1 + k_2x_2 + \cdots + k_nx_n + v \tag{5}$$

becomes in the transformed system

$$u = \underbrace{K\hat{T}}_{\hat{K}}\hat{\mathbf{x}} + v = \hat{k}_1\hat{x}_1 + \hat{k}_2\hat{x}_2 + \cdots + \hat{k}_n\hat{x}_n + v \tag{6}$$

Substituting Equation (6) into Equation (1) gives

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \hat{A}\hat{\mathbf{x}} + \hat{b}\left(\hat{K}\hat{\mathbf{x}} + v\right) \\
&= \left(\hat{A} + \hat{b}\hat{K}\right)\hat{\mathbf{x}} + \hat{b}v \\
&= \left[\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \hat{k}_1 & \hat{k}_2 & \cdots & \hat{k}_n \end{pmatrix}\right]\hat{\mathbf{x}} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v \\
&= \left[\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \hat{k}_1 & \hat{k}_2 & \hat{k}_3 & \cdots & \hat{k}_n \end{pmatrix}\right]\hat{\mathbf{x}} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v \\
&= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -(a_0 - \hat{k}_1) & -(a_1 - \hat{k}_2) & -(a_2 - \hat{k}_3) & \cdots & -(a_{n-1} - \hat{k}_n) \end{pmatrix}\hat{\mathbf{x}} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v (7)
\end{aligned}
$$

In Equation (7), the matrix $\hat{A}+\hat{b}\hat{K}$ is in companion form, and thus the actual closed loop characteristic polynomial is

$$\chi_{cl}^{act}(\lambda) = \lambda^n + (a_{n-1} - \hat{k}_n)\lambda^{n-1} + \cdots + (a_1 - \hat{k}_2)\lambda + (a_0 - \hat{k}_1) \qquad (8)$$

Comparing Equation (4) with Equation (8) we see that $\hat{a}_i = a_i - \hat{k}_{i+1}$ for $i = 0, 1, \ldots n-1$, and thus we get expressions for the $\hat{k}$ values:

$$
\begin{aligned}
\hat{k}_1 &= a_0 - \hat{a}_0 \\
\hat{k}_2 &= a_1 - \hat{a}_1 \\
\hat{k}_3 &= a_2 - \hat{a}_2 \\
&\vdots \\
\hat{k}_n &= a_{n-1} - \hat{a}_{n-1}
\end{aligned} \qquad (9)
$$

Finally from Equation (6), we compute

$$K = \hat{K}\hat{T}^{-1} \qquad (10)$$

3 *Ackermann*'s Formula

This is an alternative way of determining pole placement gains for single input systems. The formula is

$$K = -[0, 0, \ldots, 0, 1]U^{-1}\chi_{cl}^{des}(A) \tag{11}$$

where $U = [b, Ab, \ldots, A^{n-1}b]$ is the controllability matrix of the open-loop system, and $\chi_{cl}^{des}(A) = A^n + \hat{a}_{n-1}A^{n-1} + \cdots + \hat{a}_1 A + \hat{a}_0 I$ is the desired characteristic polynomial of the closed-loop system matrix evaluated at the open-loop system matrix.

Ackermann's Formula dose not require any coordinate changes to find the feedback matrix.

Why does the formula work?

Consider the closed-loop characteristic polynomial evaluated at the companion matrix $\hat{A}$

$$\chi_{cl}^{des}(\hat{A}) = \hat{A}^n + \hat{a}_{n-1}\hat{A}^{n-1} + \cdots + \hat{a}_1\hat{A} + \hat{a}_0 I.$$

By the *Cayley-Hamilton* theorem

$$\chi_{ol}(\hat{A}) = \hat{A}^n + a_{n-1}\hat{A}^{n-1} + \cdots + a_1\hat{A} + a_0 I = 0$$

and so

$$\begin{aligned}
\chi_{cl}^{des}(\hat{A}) &= (\hat{a}_{n-1} - a_{n-1})\hat{A}^{n-1} + (\hat{a}_{n-2} - a_{n-2})\hat{A}^{n-2} \cdots + (\hat{a}_1 - a_1)\hat{A} + (\hat{a}_0 - a_0)I \\
&= -\hat{k}_n\hat{A}^{n-1} - \hat{k}_{n-1}\hat{A}^{n-2} \cdots - \hat{k}_2\hat{A} - \hat{k}_1 I
\end{aligned} \tag{12}$$

Letting

$$\mathbf{e}_i^T \triangleq (0, 0, \ldots, 1, \ldots, 0)$$

(with 1 in the i-th position), then it can be shown that

$$\mathbf{e}_1^T\hat{A}^q = \mathbf{e}_{q+1}^T, \qquad q = 0, 1, 2, \ldots, n-1$$

Multiplying Eq (12) on the left by $\mathbf{e}_1^T$ gives

$$
\begin{aligned}
\mathbf{e}_1^T \chi_{cl}^{des}(\hat{A}) &= -\hat{k}_n \mathbf{e}_1^T \hat{A}^{n-1} - \hat{k}_{n-1} \mathbf{e}_1^T \hat{A}^{n-2} \cdots - \hat{k}_2 \mathbf{e}_1^T \hat{A} - \hat{k}_1 \mathbf{e}_1^T I \\
&= -\hat{k}_n \mathbf{e}_n^T - \hat{k}_{n-1} \mathbf{e}_{n-1}^T \cdots - \hat{k}_2 \mathbf{e}_2^T - \hat{k}_1 \mathbf{e}_1^T \\
&= [-\hat{k}_1, -\hat{k}_2, \ldots, -\hat{k}_{n-1}, -\hat{k}_n] \\
&= -\hat{K} \\
&= -K\hat{T}. \qquad (13)
\end{aligned}
$$

Since $\hat{A}^q = \hat{T}^{-1} A^q \hat{T}$ for $q \geq 0$, we have $\chi_{cl}^{des}(\hat{A}) = \hat{T}^{-1} \chi_{cl}^{des}(A) \hat{T}$ and so Eq (13) becomes (on interchanging LHS and RHS)

$$
\begin{aligned}
-K\hat{T} &= \mathbf{e}_1^T \hat{T}^{-1} \chi_{cl}^{des}(A) \hat{T} \\
\Rightarrow \quad K &= -\mathbf{e}_1^T \hat{T}^{-1} \chi_{cl}^{des}(A) \\
&= -\mathbf{e}_1^T \hat{T}^{-1} \chi_{cl}^{des}(A) \\
&= -\underbrace{\mathbf{e}_1^T \hat{U}}_{\mathbf{e}_n^T} U^{-1} \chi_{cl}^{des}(A) \\
&= -\mathbf{e}_n^T U^{-1} \chi_{cl}^{des}(A) \qquad (14) \\
&= -[0, 0, \ldots, 0, 1] U^{-1} \chi_{cl}^{des}(A).
\end{aligned}
$$