PDEs

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Chapter 1

Partial Differential Equations

Partial differential equations arise in connection with various physical and geometrical problems when the functions involved depend on two or more independent variables. These variables may be the time and one or more co-ordinates in space. We will consider some of the most important partial differential equations occurring in engineering applications. Only the simplest physical systems can be modelled by **ordinary differential equations**, whereas most problems in fluid mechanics, elasticity, heat transfer, chemical reactions, electromagnetic theory and other areas of study lead to **partial differential equations**.
We shall derive these equations from physical principles and consider methods for solving initial and boundary value problems, that is, methods for obtaining solutions of those equations corresponding to the physical situations.

1.1 Basic Concepts

An equation involving one or more partial derivatives of an (unknown) function of two or more independent variable is called a **partial differential equation**. The order of the highest derivative is called the **order** of the equation. The unknown function is often termed the *dependent variable*. Just as in the case of an ordinary differential equation, we say that a partial differential equation is **linear** if it is of the first degree in the dependent variable and its partial derivatives. If each term of such an equation contains either the dependent variable or one of its derivatives, the equation is said to be **homogeneous**, otherwise it is said to be **inhomogeneous**.

Some Important linear partial differential equations of the second order are:

1. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ (one dimensional wave equation)
2. $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ (one dimensional heat equation)

3. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (two dimensional Laplace equation)

4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y)$ (two dimensional Poisson equation)

5. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ (three dimensional Laplace equation)

6. $-\dfrac{h^2}{2m}\nabla^2\psi + v(\mathbf{x})\,\psi = ih\dfrac{\partial \phi}{\partial t}$ (Schrödinger equation)

Here c, h, m are known constants, t is the time and x, y, z are Cartesian coordinates. Equation 4. (with $f \neq 0$) is nonhomogeneous, while the other equations are homogeneous. In all these equations u is the dependent variable while x, y, z, t are the independent variables. For example in 1. and 2. $u = u(x,t)$, whereas in 3. $u = u(x,y)$ etc.

A **solution** to a partial differential equation in some region R of the space of the independent variables is a function of the independent variables which has all the partial derivatives appearing in the equation and satisfies the equation everywhere in R. Thus if the independent variables are x, y, z, t the solution would be of the form $u = u(x,y,z,t)$.

In general, the totality of solutions of a partial differential equation is very large. For example, the functions

$$u = x^2 - y^2, \qquad u = e^x \cos y, \qquad u = \ln(x^2 + y^2),$$

are all solutions of the Laplace equation 3., even though they look entirely different. We shall see later that the unique solution of a partial differential equation corresponding to a given physical problem will be obtained by the use of additional information arising from the physical situation. For example, in some cases the values of the required solution of the problem on the boundary of some domain will be given (**boundary conditions**); in other cases when the time t is one of the variables, the values of the solution at $t = 0$ will be prescribed (**initial conditions**).

We recall that if an *ordinary differential equation* is linear and homogeneous, then from known solutions further solutions can be obtained by superposition. For a homogeneous linear *partial differential equation* the situation is quite similar. In fact, the following theorem holds:

Theorem 1.1.1 *If u_1 and u_2 are any solutions of a linear homogeneous partial differential equation in some region, then $u = c_1u_1 + c_2u_2$ where c_1 and c_2 are any constants, is also a solution of that equation in that region.*

Remark In many situations we will be seeking to superpose an **infinite** number of solutions. Suppose that u_1, u_2, u_3, ... are all solutions of an equation $L[u] = 0$ (L is an operator and so $L[u] = 0$ is a PDE, for example in the Laplace equation, $L[u] = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$). Then

$$\sum_{n=1}^{\infty} a_n u_n,$$

is also a solution provided

- $L[u_1] = 0$, $L[u_2] = 0$, $L[u_3] = 0$, etc., **and**
- The infinite series is convergent and differentiable term by term as many times as is needed in the definition of the linear operator L.

1.1.1 Inhomogeneous Problems

Many problems involve an inhomogeneous equation containing a term corresponding to applied "forces" or "sources". For example, if a force $f(x,t)$ is applied to a vibrating string, the equation is inhomogeneous:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{T} f(x,t).$$

A problem may be inhomogeneous because of the boundary conditions as well as the equation itself. The criterion for a linear homogeneous boundary value problem is that if u is a solution of the equation *and* its boundary conditions, then so is any multiple of u. An example of an inhomogeneous boundary condition is a vibrating string for which the end $x = 0$ is prescribed to move in a certain way, $u(0,t) = g(t) \neq 0$. The general solution of an inhomogeneous problem is made up of any particular solution of the problem plus the general solution of the corresponding homogeneous problem, for which both the equation and boundary conditions are homogeneous. This composition of solution is very similar to the case of ordinary differential equations. When developing solution techniques for PDEs, we will mainly concentrate on homogeneous problems.

1.1.2 Pseudo or degenerate PDEs

It is important to appreciate that a PDE involving derivatives with respect to one variable only may be solved like an ordinary differential equation treating the other independent variable as parameters. For example, if $u = u(x,y)$ the equation $\frac{\partial u}{\partial x} = 0$ can be solved using ODE techniques. Essentially consider the ODE $\frac{du}{dx} = 0$ which has a solution $u =$ constant but allow this "constant" to be a function of y. Thus the solution of $\frac{\partial u}{\partial x} = 0$ is $u = a(y)$ where $a(y)$ is any function of y but does not depend on x. Checking back shows that $u = a(y)$ does satisfy the equation $\frac{\partial u}{\partial x} = 0$.

Example 1.1.1 *Solve the following PDEs:*

$$\textit{(i)} \quad \frac{\partial^2 u}{\partial y^2} = 0, \qquad \textit{(ii)} \quad \frac{\partial u}{\partial x} = x, \qquad \textit{(iii)} \quad \frac{\partial u}{\partial x} + u = 0, \qquad \textit{(iv)} \quad \frac{\partial^2 u}{\partial y^2} + u = 0,$$

where $u = u(x,y)$.

Note: A very common notation for partial derivatives is to use subscripts. Thus $\frac{\partial u}{\partial x} \equiv u_x$; $\frac{\partial^2 u}{\partial x^2} \equiv u_{xx}$ etc. In fact ODEs can also be written in this way. The disadvantage of the subscript notation is

that it does not explicitly distinguish between partial and ordinary derivatives. If $u = u(x, y)$ then $u_x \equiv \frac{\partial u}{\partial x}$ but if $u = u(x)$ then $u_x \equiv \frac{du}{dx}$.

1.2 Classification of second order linear PDEs

We consider the case of linear second order PDEs with two independent variables. This simplifies matters and much of the reasoning can be generalised to equations with more independent variables. We consider a function $u(x, y)$ to be evaluated in some region of the xy-plane (in which $u(x, y)$ satisfies some PDE). The partial differential equation will need to be supplemented by boundary conditions of some sort; we assume these involve values of u and/or its derivatives on the curve which encloses the region within which we are trying to solve the equation. There are three common types of boundary condition.

1. *Dirichlet* conditions: u is specified at each point on the boundary.

2. *Neumann* conditions: $\frac{\partial u}{\partial n}$, the normal derivative, i.e. the directional derivative in the direction of the normal to the bounding curve is given at every point on the boundary. (Recall $\frac{\partial u}{\partial n} = \hat{\mathbf{n}} \cdot \nabla u$, where $\hat{\mathbf{n}}$ is the unit normal vector at each point on the boundary).

3. *Cauchy* conditions: u and $\frac{\partial u}{\partial n}$ are given at every point on the boundary.

Other possibilities include Robin conditions where a linear combination of u and $\frac{\partial u}{\partial n}$ is given on the boundary. Also possible is that one type of condition, e.g. Dirichlet, is given on one part of the boundary and a different type on the remainder. However, we restrict attention to the three basic types.

By analogy with second order ODEs, we would expect that Cauchy conditions along a line would be the most natural.

1.2.1 Characteristics of PDEs

We now classify linear second order PDEs into three types: **Parabolic**, **Hyperbolic** and **Elliptic**. They lead to families of characteristic curves for each type which can help to solve the PDEs.

We start by considering the differential equation

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + 2B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} = f\left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \tag{1.1}$$

Let us suppose the boundary curve is given *parametrically* by the equations

$$\mathbf{r}(s) = (x(s), y(s)),$$

where s is arclength along the boundary. We shall suppose that we are given $u(x,y)$ and $N(x,y) = \frac{\partial u}{\partial n}$ along the curve and we thus know both of these as functions of s, i.e. $u(s)$ and $N(s)$. As s is the arclength the components of the unit tangent vector $\hat{\mathbf{t}}$ to the curve are $\left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ and so the components of the unit normal $\hat{\mathbf{n}}$ to the curve are $\left(-\frac{dy}{ds}, \frac{dx}{ds}\right)$ (because $\hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = 0$). Then

$$\frac{\partial u}{\partial n} = N(s) = \hat{\mathbf{n}} \cdot \nabla u = -\frac{\partial u}{\partial x}\frac{dy}{ds} + \frac{\partial u}{\partial y}\frac{dx}{ds}. \tag{1.2}$$

We can also write

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}, \tag{1.3}$$

which can be solved with (1.2) (using the fact that $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$) and this gives:

$$\frac{\partial u}{\partial x} = -N(s)\frac{dy}{ds} + \frac{du}{ds}\frac{dx}{ds} \tag{1.4}$$

$$\frac{\partial u}{\partial y} = N(s)\frac{dx}{ds} + \frac{du}{ds}\frac{dy}{ds}. \tag{1.5}$$

The trouble comes with second derivatives. There are **three** of these to be found

$$\frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial x \partial y}, \qquad \frac{\partial^2 u}{\partial y^2}.$$

We thus need three equations to determine these three unknown functions. Two equations are found by differentiating the (now known) first derivatives along the boundary using the chain rule

$$\frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{dx}{ds} + \frac{\partial^2 u}{\partial x \partial y}\frac{dy}{ds} \tag{1.6}$$

$$\frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2 u}{\partial x \partial y}\frac{dx}{ds}. \tag{1.7}$$

A third equation is provided by the original differential equation (1.1). The three equations (1.1), (1.6) and (1.7) are written in matrix form in the following manner:

$$\begin{bmatrix} \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \\ A & 2B & C \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} \frac{d}{ds}(u_x) \\ \frac{d}{ds}(u_y) \\ f(x,y,u_x,u_y) \end{bmatrix}.$$

[This is the familiar matrix equation $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ are known and we wish to find $\mathbf{x}$. Recall that $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A has an inverse, i.e. if and only if A^{-1} exists, i.e if and only if $\det A \neq 0$].

Thus these three inhomogeneous equations can be solved for the second partial derivatives of u *unless* the determinant of the coefficients vanishes, i.e. unless

$$\det \begin{bmatrix} \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & \frac{dx}{ds} & \frac{dy}{ds} \\ A & 2B & C \end{bmatrix} = 0,$$

or on expanding the determinant along the **bottom row**

$$A(x,y)\left(\frac{dy}{ds}\right)^2 - 2B(x,y)\frac{dx}{ds}\frac{dy}{ds} + C(x,y)\left(\frac{dx}{ds}\right)^2 = 0, \tag{1.8}$$

or alternatively

$$A(x,y)\left(\frac{dy}{dx}\right)^2 - 2B(x,y)\frac{dx}{dx}\frac{dy}{dx} + C(x,y)\left(\frac{dx}{dx}\right)^2 = 0. \tag{1.9}$$

Recall that the boundary curve was given parametrically by $\mathbf{r} = \mathbf{r}(s) = (x(s), y(s))$. Equation (1.8) (or 1.9) defines two directions at every point in space, namely the direction given by the vector $\left(\frac{dx}{ds}, \frac{dy}{ds}\right)$. As (1.8) is quadratic, in general it contains two solutions for $\frac{dx}{ds}$ and $\frac{dy}{ds}$, i.e. we can associate two directions with every point in space. These are called the *characteristic directions* at each point. Curves in the xy-plane whose tangents at each point lie along the characteristic directions are called **characteristics** of the PDE. To map out the characteristics for a particular equation, it is simplest to solve (1.9) for $\frac{dy}{dx}$ and then to solve the two resulting differential equations to give y as a function of x.

Returning to the equation (1.9) for the characteristics we see that this has solution for $\frac{dy}{dx}$ given by:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}, \tag{1.10}$$

where A, B, C can be functions of x and y. The classification is as follows:

- If the characteristics are to be real curves, we clearly must have $B^2 > AC$. PDEs obeying this condition are called **hyperbolic** and have two sets of characteristics, e.g. the wave equation $u_{tt} = c^2 u_{xx}$.
- If $B^2 - AC = 0$ the equation is said to be **parabolic** and has one set of characteristic curves, e.g. the heat equation $u_t = c^2 u_{xx}$.
- If $B^2 - AC < 0$ the equation is **elliptic** and has no (real) characteristics, e.g. the Laplace equation $u_{xx} + u_{yy} = 0$.

Boundary Conditions

Let us discuss the choice of boundary conditions which is appropriate for each of the three types of equation, beginning with the hyperbolic equation. We have seen above that, generally speaking, Cauchy conditions along a curve which is not characteristic are sufficient to specify the solution near that curve. A useful picture for visualising the role of the characteristics and boundary conditions is obtained by thinking of the characteristics as curves along which partial information about the solution propagates. The meaning of this statement and the way in which it works are most easily understood with the aid of an elementary example.

Example 1.2.1 *Consider the simplest hyperbolic equation, having $A = c^2$, $B = 0$, $C = -1$, where c is a known constant. This is the one dimensional wave equation*

$$u_{tt} = c^2 u_{xx}, \tag{1.11}$$

for which the equation of the characteristics (1.8) (identify x with x and t with y) is

$$c^2 \left(\frac{dt}{ds}\right)^2 - \left(\frac{dx}{ds}\right)^2 = 0,$$

or

$$\left(\frac{dx}{dt}\right)^2 = c^2 \qquad \Longrightarrow \qquad \frac{dx}{dt} = \pm c.$$

Thus the characteristics are obtained by integrating these last two equations yielding solutions

$$x - ct = \xi \quad \textit{(constant)}, \qquad x + ct = \eta \quad \textit{(constant)}, \tag{1.12}$$

where ξ and η are arbitrary constants. As these vary over a range of values, (1.12) maps out a family of characteristics in xt-space (which are straight lines in this instance), see Figure 1.1.

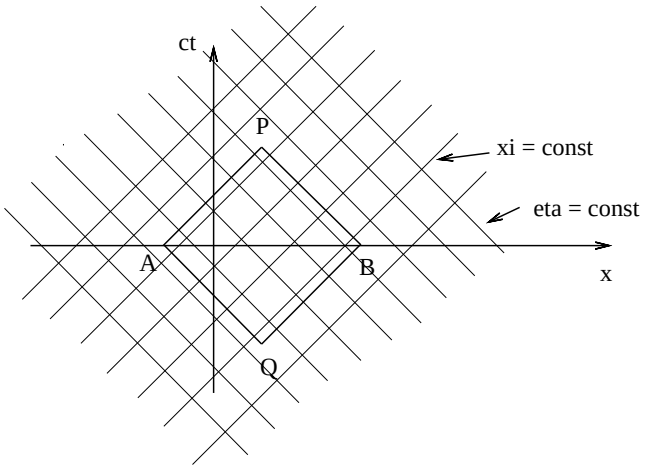

Figure 1.1: *Characteristics for the one-dimensional wave equation.*

The characteristics form a "natural" set of coordinates for a hyperbolic equation. For example, if we transform (1.11) to the new coordinates ξ and η defined in (1.12) using the chain rule for partial differentiation, we obtain the equation in so called normal or canonical form:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \tag{1.13}$$

This can be integrated directly yielding the solution

$$u = g(\xi) + h(\eta) = g(x - ct) + h(x + ct), \tag{1.14}$$

where g and h are (sufficiently well behaved) functions.

Now, if we know $u(x,0) = a(x)$ *and its normal derivative at* $t = 0$*:* $N(x) = \mathbf{k} \cdot \nabla u = (0,1) \cdot (\frac{\partial u}{\partial x}, \frac{1}{c}\frac{\partial u}{\partial t}) = \frac{1}{c}\frac{\partial u}{\partial t}(x,0) = b(x)$*, where* $a(x)$ *and* $b(x)$ *are known functions, along the line segment* AB *of Figure 1.1, we can find the individual functions* $g(\xi)$ *and* $h(\eta)$ *as follows:*

$$u(x,0) = a(x) = g(x) + h(x)$$

and

$$\frac{1}{c}\frac{\partial u}{\partial t}(x,0) = b(x) = -g'(x) + h'(x),$$

which on integrating w.r.t. x *gives*

$$\int b(x)\,dx = -g(x) + h(x) + K,$$

where K *is an arbitrary integration constant. We now have two equations for the two unknown functions* $g(x)$ *and* $h(x)$ *which we solve to find*

$$g(x) = \frac{1}{2}a(x) - \frac{1}{2}\int b(x)\,dx + \frac{1}{2}K$$
$$h(x) = \frac{1}{2}a(x) + \frac{1}{2}\int b(x)\,dx - \frac{1}{2}K.$$

These are then substituted back into (1.14). The arbitrary constant associated with the integral is of no importance since it cancels in this sum.

The results obtained for the simple case above hold generally for hyperbolic equations. We summarise in the table below the types of boundary conditions appropriate for different equations. This table is not exhaustive but gives a general idea of what type the boundary conditions look like for a particular problem. A very useful way of checking if a problem is well posed (i.e. that the boundary conditions are sufficient to determine the solution) is to consider a physical process represented by the problem and intuitively decide whether the physical problem is well posed i.e. capable of solution.

Equation	*Condition*	*Boundary*
Hyperbolic	Cauchy	Open
Elliptic	Dirichlet or Neumann	Closed
Parabolic	Dirichlet or Neumann	Open

Recall that Cauchy conditions physically correspond to a given displacement and velocity at time $t = 0$, i.e. specifying u and $\frac{\partial u}{\partial t}$. Dirichlet conditions correspond to a given displacement on a curve in the (x,t) plane, i.e. specifying u. Neumann conditions correspond to stating a velocity on a curve in the (x,t) plane, i.e. specifying $\frac{\partial u}{\partial t}$.

1.2.2 Qualitative behaviour of elliptic, parabolic and hyperbolic equations

Elliptic equations (e.g. $\nabla^2 u = 0$)

These are generally associated with equilibrium phenomena when transient (time dependent) effects have died away. Elliptic problems are thus not time dependent and are usually boundary value problems. For these types of problems either Dirichlet, Neumann or mixed conditions on a closed boundary are necessary to get a well-posed problem.

Elliptic problems sometimes result from the long time solution of parabolic problems. Consider a thin rectangular plate with a given initial temperature distribution whose edges are each held at some particular temperature. If we wait "long enough", it is possible that the temperature in the plate will settle down to a time-independent (steady) value described by the Laplace equation with boundary conditions. Initially the problem would have been described by the heat equation $\frac{\partial u}{\partial t} = c^2 \nabla^2 u$ but *if the problem has a steady state solution*, then $\frac{\partial}{\partial t} \equiv 0$ and we reduce to Laplace's equation, the initial condition having been forgotten. Resorting to intuitive physical arguments is a useful way of reasoning whether or not a problem has a steady state solution.

To gain an intuitive understanding consider an elliptic equation in a finite rectangular region. Identify the elliptic equation as a steady state heat flow in a 2D rectangular plate where $u(x, y)$ is the temperature. We emphasize that a steady state must have been reached: u varies depending upon where you look on the plate but each $u(x, y)$ is constant in time. Boundary conditions in u must be translated into conditions in terms of the temperature at the edges. If $u = 0$ along some edge, then this edge of the plate is imagined as having its temperature held at 0, e.g. by placing a strip of ice along it.

Parabolic equations (e.g. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$)

Parabolic equations describe evolution type processes i.e. processes which involve time. Typically a parabolic equation represents a diffusive or "spreading out" phenomenon where, for example a concentrated patch of dye in water "smears out" until it is distributed evenly everywhere. Another typical example is heat flow in a rod where one point on the rod has a "hot-spot" initially but as the diffusion process evolves, the hot spot smoothes out until finally the temperature is everywhere the same (if the boundary conditions will allow this). Parabolic problems are initial-boundary value problems in that they require both initial and boundary conditions. Generally speaking we require Dirichlet or Neumann conditions on an open boundary.

To gain an intuitive understanding consider a parabolic equation in a one dimensional strip of finite length l. Identify this with time dependent heat flow in a thin (metal) rod of finite length. The initial condition $u(x, 0) = f(x)$, describes the temperature as a function of distance (x) along the rod at time $t = 0$. Boundary conditions describe the temperatures in the rod at the ends (which are

held fixed in time). For example, if $u(0,t) = 0$, then one end of the rod can be thought of as being emersed in iced water or ice (zero degrees). The mathematical problem requires a solution $u(x,t)$ for all x in $[0,l]$ for all $t \geq 0$. Physically this corresponds to finding the temperature at every point (x) in the rod for all times in the future (t).

Hyperbolic equations (e.g. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$)

These again describe evolution type phenomenon and again both initial and boundary conditions are required in general. Hyperbolic equations usually represent wave-like phenomena whereby some initial condition or signal (or "wave") is seen to propagate through space either maintaining its original form or changing shape and/or velocity as it does so but remaining generally recognizable as the original signal or "wave".

To gain an intuitive understanding consider a hyperbolic equation in a long strip of length l. Identify this with waves passing down a taut, thin, long string of length l. The initial conditions describe an (initial) disturbance, e.g. plucking the string at some point. This "wave" will then pass along the string. What other conditions are necessary to ensure a complete description of the evolution of the disturbance of the string? Obviously if the ends of the string are fixed then conditions like $u(0,t) = 0$ and $u(l,t) = 0$ are required. In addition to the already mentioned displacement, we also require the initial velocity in the string. It is possible to have different situations with the same initial displacement but different initial velocity.

1.2.3 An overview of solution method for linear second order PDEs

1. **Separation of variables**. Usually require a constant coefficient equation. The basic idea is to use the homogeneous (zero) boundary conditions to find the eigenfunctions. Then writing the solution as a linear combination of the eigenfunctions, use the inhomogeneous boundary (or initial) condition to solve for the superposition constants. If there is more than one inhomogeneous condition, superposition of problems exploiting the linearity of the equation may be necessary.

2. **Integral transforms**. Again usually only applicable to constant coefficient equations. The basic philosophy is to transform "out" one independent variable reducing to a simpler problem (often an ODE). The particular transform used often depends on the boundary conditions and the ranges of the independent variables. For example, if time $\in [0,\infty)$ a Laplace transform may be used.

3. **Green's functions**. A very general method for solving PDEs (and ODEs) yielding solutions in the form of an integral expression.

4. **Complex variable techniques**. Laplace's (and Poisson's) equations are fundamental to many physical processes. For example $\nabla^2 T = 0$ describes the steady state heat distribution in an object. In complex variable theory, a basic property of **any** analytic function $f(z) = u(x,y) + iv(x,y)$ is that both the real and imaginary parts of $f(z)$ satisfy Laplace's equation i.e. $\nabla^2 u = 0$ and $\nabla^2 v = 0$. Because of this property it is possible to reduce many problems involving Laplace's equation to exercised in complex variable theory.

5. **Numerical techniques** If all else fails, we can resort to numerical techniques, e.g. finite differences, finite element method, finite volume method, boundary elements.

1.3 Solutions to some elementary linear ODEs

We take $y = y(x)$ to be the dependent variable and x to be the independent variable.

1.3.1 First-order equation: variables separable

This has the basic form:

$$h(x)k(y)\frac{dy}{dx} = -f(x)g(y), \tag{1.15}$$

where $f(x)$, $h(x)$ are given functions of x, and $g(y)$, $k(y)$ are given functions of y. If the equation has this form, the variables can be separated by dividing through by $h(x)g(y)$ to obtain

$$\frac{k(y)}{g(y)}\frac{dy}{dx} = -\frac{f(x)}{h(x)}.$$

On multiplying across by dx we have separated the variables and the equation can be solved by straightforward integration:

$$\frac{k(y)}{g(y)}dy = -\frac{f(x)}{h(x)}dx \qquad \Longrightarrow \qquad \int \frac{k(y)}{g(y)}dy = -\int \frac{f(x)}{h(x)}dx,$$

as in principle both of these integrations can be performed.

Example 1.3.1 *Find the general solution of*

$$\frac{dG}{dt} = -kG, \tag{1.16}$$

where k is a known constant.

1.3.2 First-order linear (in y) equation

This is more general than the last situation and has the form:

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1.17}$$

where $P(x)$ and $Q(x)$ are known functions. This equation can be solved by multiplying both sides by an integrating factor which is of the form $\exp(\int P(x)dx)$. (The equation must be in **exactly** the same form as (1.17), i.e. the coefficient of $\frac{dy}{dx} = 1$). On multiplying by the integrating factor we obtain

$$\frac{d}{dx}\left[y\exp\left(\int P(x)dx\right)\right] = Q(x)\exp\left(\int P(x)dx\right), \tag{1.18}$$

which can be integrated directly.

Example 1.3.2 *Find the general solution of*

$$\frac{dy}{dx} + \frac{y}{x} = 2x.$$

1.3.3 Second order linear equations

The equation

$$\frac{d^2y}{dx^2} = -n^2y, \tag{1.19}$$

where n is a given real constant, has general solution

$$y = A\cos nx + B\sin nx, \tag{1.20}$$

where A and B are arbitrary constants to be determined from the boundary conditions.

The equation

$$\frac{d^2y}{dx^2} = n^2y, \tag{1.21}$$

where n is a given real constant, has general solution $y = A\cosh nx + B\sinh nx$, or equivalently

$$y = Ce^{nx} + D^{-nx}, \tag{1.22}$$

where A, B, C, D are arbitrary constants to be determined from the boundary conditions.

1.3.4 Basic Fourier theory

We summarise here the basic results of Fourier series theory.

Fourier series theorem

If $f(x)$ and $\frac{df}{dx}$ are piecewise continuous on the interval $-l \le x \le l$ and $f(x)$ is defined outside the interval $[-l, l]$ so that it is periodic with period $2l$, then f has a Fourier series

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}\right), \tag{1.23}$$

with

$$A_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad B_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx. \tag{1.24}$$

Even and odd functions

If $f(x)$ is an **even** function, i.e. $f(x) = f(-x)$ and satisfies the conditions of the Fourier theorem then it can be expressed as a Fourier *cosine* series:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}, \qquad \text{with} \qquad A_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx. \tag{1.25}$$

If $f(x)$ is an **odd** function, i.e. $f(x) = -f(-x)$ and satisfies the conditions of the Fourier theorem then it can be expressed as a Fourier *sine* series:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}, \qquad \text{with} \qquad B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \tag{1.26}$$

Orthogonality of sine and cosine

Full range:

$$\int_{-l}^{l} \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = \begin{cases} 0 & m \neq n \\ l & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = \begin{cases} 0 & m \neq n \\ l & m = n \neq 0 \\ 2l & m = n = 0 \end{cases}$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = 0 \quad \text{all } m,\, n.$$

Half range:

$$\int_{0}^{l} \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = \begin{cases} 0 & m \neq n \\ \frac{l}{2} & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

$$\int_{0}^{l} \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx = \begin{cases} 0 & m \neq n \\ \frac{l}{2} & m = n \neq 0 \\ 2l & m = n = 0. \end{cases}$$

In solving PDEs using separation of variables, we will often have cause to express a function $f(x)$ defined on some range $[0, l]$, e.g. on a conducting bar of length l, in a Fourier series. As we are only concerned with the behaviour of f inside $[0, l]$ we can define it however we like outside $[0, l]$. Typically we might define f to be odd (or even) and periodic with period $2l$ and then express it in the form of a fourier half range sine (or cosine) series using the above results.

1.4 Vibrating String (1D Wave equation)

In this section we consider the transverse vibrations of an elastic string. We begin by deriving a PDE governing the vibrations and then show how solutions to some typical problems can be obtained.

1.4.1 Derivation of the PDE for the vibrating string

As a first important partial differential equation, let us derive the equation governing small transverse vibrations of an elastic string, which is stretched to length l and then fixed at the end points. Suppose that the string is distorted and then at a certain instant, say, $t = 0$, it is released and allowed to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at time $t > 0$, see Figure 1.2.

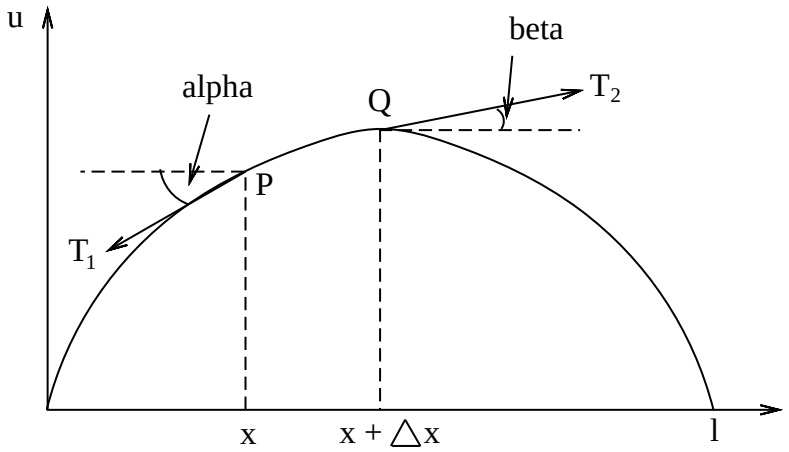

Figure 1.2: *Vibrating string (1D).*

When deriving a differential equation corresponding to a given physical problem, we usually have to make simplifying assumptions in order that the resulting equation does not become too complicated. In our present case we make the following assumptions:

1. The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.

2. The tension caused by stretching the string before fixing it at the end points is so large that the action of the gravitational force on the string can be neglected.

3. The motion of the string is a small transverse vibration in the vertical plane, that is, each particle of the string moves strictly vertically, and the deflection and the slope at any point of the string are small in absolute value. (**Note**: a lot of simplification in applied mathematics is obtained by exploiting the smallness of some quantity).

These assumptions are such that we may expect that the solution $u(x,t)$ of the differential equation to be obtained will reasonably well describe small vibrations of the physical "nonidealised" string of small homogeneous mass under large tension.

To obtain the differential equation we consider the forces acting on a small portion of the string (Figure 1.2). Since the string does not offer resistance to bending, the tension is tangential to the curve of the string at each point. Let T_1 and T_2 be the tensions at the end points P and Q of that portion. Since there is no motion in the horizontal direction, the horizontal components of the tension must be constant. Using the notation shown in Figure 1.2, we thus obtain

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{constant}. \tag{1.27}$$

In the vertical direction we have two forces, namely the vertical components $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of T_1 and T_2: here the minus sign appears because the component at P is directed downwards. By Newton's second law the resultant of those two forces is equal to the mass $\rho\Delta x$ of the portion times the acceleration u_{tt}, evaluated at some point between x and $x+\Delta x$: here ρ is the mass of the undeflected string per unit length, and Δx is the length of the portion of the undeflected string. Hence

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x\, u_{tt}.$$

If this equation is divided by T and the expressions in (1.27) are substituted we find

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho\Delta x\, u_{tt}}{T}. \tag{1.28}$$

Now $\tan\alpha$ and $\tan\beta$ are the slopes of the curve of the string at x and $x+\Delta x$, that is

$$\tan\alpha = \left.\frac{\partial u}{\partial x}\right|_x, \qquad \tan\beta = \left.\frac{\partial u}{\partial x}\right|_{x+\Delta x},$$

where the subscripts here mean evaluated at the point in question.

Here we have to write *partial derivatives* because u depends also on t. Dividing (1.28) by Δx, we deduce that

$$\frac{1}{\Delta x}\left(\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_x\right) = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}.$$

If we let Δx approach zero, we obtain the linear homogeneous partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{or} \qquad u_{tt} = c^2 u_{xx},$$

where $c^2 = \frac{T}{\rho}$. This is the so-called **one-dimensional wave equation**, which governs our problem. It is parabolic as we showed in § 4.2. The notation c^2 for the physical constant $\frac{T}{\rho}$ has been chosen to indicate that this constant is positive.

1.4.2 Separation of Variables (for the vibrating string)

Formulation of the problem

We have seen that the vibrations of an elastic string are governed by the one-dimensional wave equation (hyperbolic)

$$u_{tt} = c^2 u_{xx}, \tag{1.29}$$

where $u(x,t)$ is the deflection of the string. Consider the case where the string is fixed at the ends $x = 0$ and $x = l$, so we have the two **boundary conditions**

$$u(0,t) = 0, \quad u(l,t) = 0, \qquad \text{for all } t. \tag{1.30}$$

The form of the motion of the string will depend on the initial deflection (deflection at $t = 0$) and on the initial velocity (velocity at $t = 0$). Denoting the initial deflection by $f(x)$ and the initial velocity by $g(x)$, we thus obtain the two **initial conditions**

$$u(x,0) = f(x), \tag{1.31}$$

and

$$\frac{\partial u}{\partial t} = g(x) \quad \text{when} \quad t = 0. \tag{1.32}$$

Our problem is now to find the solution of (1.29) satisfying the conditions (1.30)-(1.32).

Solution of the formulated problem

We shall proceed step by step as follows:
First Step: By applying the so-called *product method*, or *method of separating variables*, we shall obtain two ordinary differential equations.
Second Step: We shall determine solutions of those equations that satisfy the boundary conditions.
Third Step: Those solutions will be composed so that the result will be a solution of the wave equation $u_{tt} = c^2 u_{xx}$ which also satisfies the given initial conditions.

First Step:
The product method yields solutions of the equation $u_{tt} = c^2 u_{xx}$ of the form

$$u(x,t) = F(x)G(t),$$

which are a product of two functions, each depending only on one of the variables x and t. By differentiating this equation we obtain

$$u_{tt} = F\ddot{G} \quad \text{and} \quad u_{xx} = F''G,$$

where dots denote derivatives with respect to t and primes denote derivatives with respect to x. By inserting this into (1.29) we have

$$F\ddot{G} = c^2 F'' G.$$

We now divide across by FG (**Note**: this is the usual trick applied at this point) and follow this by dividing across by c^2 thus finding (for all x and t):

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}.$$

The expression on the left involves functions depending only on t while the expression on the right involves functions depending only on x. Hence both expressions must be equal to a constant, say k. [if the expression on the left is not constant, then changing t will presumably change the value of this expression but certainly not that on the right, since the latter does not depend on t. Similarly, if the expression on the right is not constant, changing x will presumably change the value of this expression but certainly not that on the left]. Therefore,

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k.$$

This immediately yields the two **ordinary** linear differential equations

$$F'' - kF = 0, \tag{1.33}$$

and

$$\ddot{G} - c^2 kG = 0. \tag{1.34}$$

In these equations, k is still arbitrary. Note that they are ODEs because $F = F(x)$ and $G = G(t)$. Before actually determining the value of k, we will first narrow down the search by finding its sign.

Second Step:

We will now determine solutions F and G of (1.33) and (1.34) so that $u = FG$ satisfies the boundary conditions in (1.30), that is

$$u(0,t) = F(0)G(t) = 0, \qquad u(l,t) = F(l)G(t) = 0 \qquad \text{for all } t.$$

Clearly, if $G = 0$, then $u = 0$, which is on no interest. Thus $G \neq 0$ and therefore

$$F(0) = 0; \qquad F(l) = 0. \tag{1.35}$$

For $k = 0$ the general solution of (1.33) is $F = ax + b$, and from the conditions in (1.35) we obtain $a = b = 0$. Hence, $F = 0$, which is of no interest because then $u = 0$.

For positive k, i.e. $k = \mu^2$, the general solution of (1.33) is (see (1.22))

$$F = Ae^{\mu x} + Be^{-\mu x},$$

and from (1.35) we obtain $F = 0$, as before. Hence we are left with the possibility of choosing k negative, say $k = -p^2$. Then (1.33) takes the form

$$F'' + p^2 F = 0,$$

and the general solution is (see (1.20))

$$F(x) = C\cos px + D\sin px.$$

From this and (1.35) we have

$$F(0) = C = 0 \qquad \text{and} \qquad F(l) = D\sin pl = 0.$$

We must take $D \neq 0$ since otherwise $F = 0$. Hence $\sin pl = 0$, that is,

$$pl = n\pi \quad \text{i.e. } p = n\pi/l \quad \text{where } n \text{ is any integer } (0, \pm 1, \pm 2, ...) \tag{1.36}$$

We thus obtain infinitely many solutions $F(x) = F_n(x)$ where

$$F_n(x) = D_n \sin\frac{n\pi x}{l} \quad n = 1, 2, 3, ..., \tag{1.37}$$

which satisfy (1.35) and the constants D_n are as yet undetermined. [For negative values of n we obtain essentially the same solutions, except for a minus sign, because $\sin(-x) = -\sin(x)$].
Then k is now restricted to $k = -p^2 = -(n\pi/l)^2$, resulting from (1.36). For these values of k the equation (1.34) takes the form

$$\ddot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = cn\pi/l.$$

The general solution is

$$G_n(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t.$$

Hence the function $u_n(x,t) = F_n(x)G_n(t)$ are given by

$$u_n(x,t) = (B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\, D_n \sin\frac{n\pi x}{l}, \qquad n = 1, 2, 3....$$

Noting the fact that the constants are arbitrary up to this point, we can redefine $B_n D_n = B_n$ (or any other letter) and $B_n^* D_n = B_n^*$ and rewrite the general solution as

$$u_n(x,t) = (B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\, \sin\frac{n\pi x}{l}, \qquad n = 1, 2, 3....$$

Third Step:
Clearly, a single solution, $u_n(x,t)$ will, not in general satisfy the initial conditions (1.31) and (1.32). Now, since the one-dimensional wave equation is linear and homogeneous, it follows from Theorem

1.1.1 that the sum of finitely many solutions u_n is a solution of the original equation (1.29). To obtain a solution that satisfies (1.31) and (1.32), we consider the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi x}{l}. \tag{1.38}$$

From this and (1.31) it follows that

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = f(x). \tag{1.39}$$

Hence, in order that the infinite series equation (1.38) satisfies the initial condition (1.31), the coefficients B_n must be chosen so that $u(x,0)$ becomes a half-range expansion of $f(x)$, namely, the Fourier sine series of $f(x)$; that is

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx, \quad n = 1, 2, 3.... \tag{1.40}$$

Similarly, by differentiating (1.38) with respect to t and using (1.32) we find

$$\begin{aligned} \frac{\partial u}{\partial t}\Big|_{t=0} &= \Big(\sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{l} \Big)_{t=0} \\ &= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{l} = g(x). \end{aligned}$$

Hence, in order that (1.38) satisfies the (1.32), the coefficient B_n^* must be chosen so that, for $t = 0$, $\partial u/\partial t$ becomes the Fourier sine series of $g(x)$; thus

$$B_n^* \lambda_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx, \quad n = 1, 2, 3...$$

or, since $\lambda_n = \frac{cn\pi}{l}$

$$B_n^* = \frac{2}{cn\pi} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx, \quad n = 1, 2, 3.... \tag{1.41}$$

Thus it follows that

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi x}{l},$$

with coefficients B_n and B_n^* given by (1.40) and (1.41), is a solution of $u_{tt} = c^2 u_{xx}$ that satisfies the boundary and initial conditions.

Example 1.4.1 *Find the solution of the wave equation (1.29) corresponding to the triangular initial deflection*

$$f(x) = \begin{cases} \frac{2kx}{l}, & when\ 0 < x < \frac{l}{2} \\ \frac{2k(l-x)}{l}, & when\ \frac{l}{2} < x < l, \end{cases}$$

and initial velocity zero. Since $g(x) = 0$, we have $B_n^* = 0$ in (1.41) and we use basic Fourier theory to solve for the B_n's giving

$$B_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}, \qquad n = 1, 2, 3, ...$$

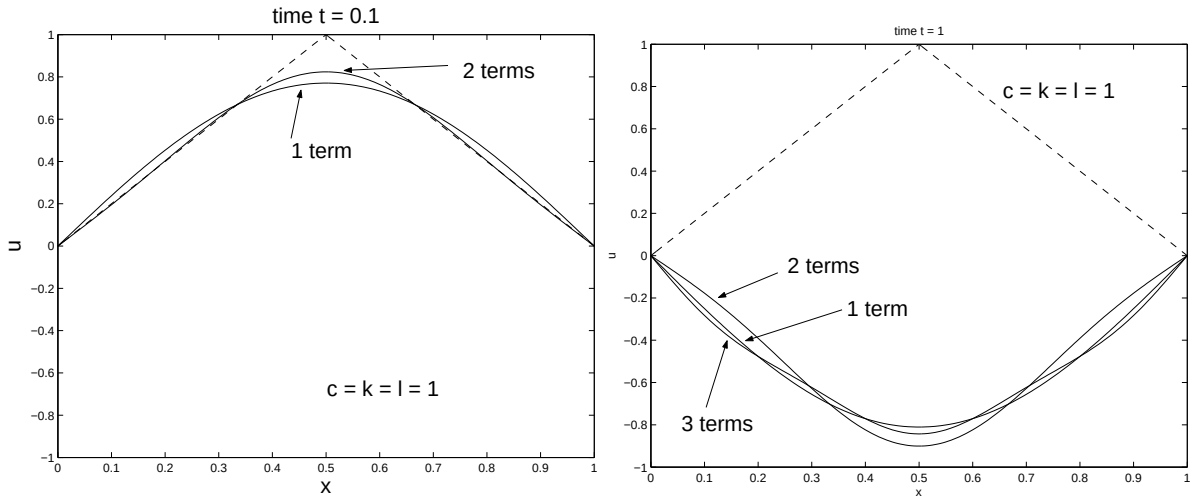

Figure 1.3: *Solution at* $t = 0.1$ *(left) and* $t = 1$ *(right). The dashed line shows the initial (triangular) condition.*

Thus (1.38) takes the form

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{8k}{n^2\pi^2} \sin\frac{n\pi}{2} \cos\frac{cn\pi t}{l} \sin\frac{n\pi x}{l} \\ &= \frac{8k}{\pi^2}\left(\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} + \dots\right). \end{aligned}$$

The results for two times are shown in Figure 1.3. The left plot has $t = 0.1$ *and shows both one and two terms in the sum (for more terms we do not observe a difference in the solution). The right plot has* $t = 1$ *with up to three terms in the sum.*

1.4.3 D'Alembert's solution of the wave equation for an infinite string

The solutions in this section were first found by D'Alambert. Strictly speaking, they do not fit in with our present investigation of the method of separation of variables but it is an alternative way of solving the wave equation on an **infinite** string subject to **Cauchy** conditions. (We already did this using separation of variables for the case of a **finite** string in the last section).

To investigate the wave equation (1.29), i.e. $u_{tt} = c^2 u_{xx}$, $c = \frac{T}{\rho}$, we introduce new independent variables (in fact these can be derived using the characteristic theory we introduced when classifying second order PDEs in § 4.2) given by

$$\eta = x + ct, \qquad \xi = x - ct. \tag{1.42}$$

Then u becomes a function of η and ξ, and the derivatives in (1.42) can be expressed in terms of derivatives with respect to η and ξ by using the chain rule. Denoting partial derivatives by

subscripts, we see from (1.42) that $\eta_x = 1$ and $\xi_x = 1$ and therefore transforming from $u(x,t)$ to $w(\eta,\xi)$, we find

$$u_x = w_\eta \eta_x + w_\xi \xi_x = w_\eta + w_\xi.$$

By applying the chain rule to the right hand side we find

$$u_{xx} \equiv (w_\eta + w_\xi)_x = (w_\eta + w_\xi)_\eta \eta_x + (w_\eta + w_\xi)_\xi \xi_x.$$

Since $\eta_x = 1$ and $\xi_x = 1$, this becomes

$$u_{xx} = w_{\eta\eta} + 2w_\eta w_\xi + w_{\xi\xi}.$$

It is customary to rewrite this using the same dependent variable on both sides of the equation i.e.

$$u_{xx} = u_{\eta\eta} + 2u_\eta u_\xi + u_{\xi\xi}.$$

The u_{tt} derivative in the wave equation is transformed by the same procedure, and the result is

$$u_{tt} = c^2\big(u_{\eta\eta} - 2u_\eta u_\xi + u_{\xi\xi}\big).$$

By inserting these two results into $u_{tt} = c^2 u_{xx}$ we obtain

$$u_{\eta\xi} = \frac{\partial^2 u}{\partial\eta\partial\xi} = 0. \tag{1.43}$$

We may integrate this equation with respect to η, finding

$$\frac{\partial u}{\partial\xi} = h(\xi),$$

where $h(\xi)$ is an arbitrary function of ξ. Integrating this with respect to ξ, we have

$$u = \int h(\xi)\, d\xi + \phi(\eta),$$

where $\phi(\eta)$ is an arbitrary function of η. Since the integral is a function of ξ, say $\psi(\xi)$, the solution u is of the form

$$u = \phi(\eta) + \psi(\xi).$$

Because of (1.42) we may write

$$u(x,t) = \phi(x + ct) + \psi(x - ct). \tag{1.44}$$

This is known as **d'Alembert's solution** of the wave equation.

The functions ϕ and ψ are arbitrary up to this point, i.e. any sufficiently smooth functions of their respective arguments will satisfy (1.43) and hence the PDE. The arbitrary functions can be determined from the initial conditions just as we could use the boundary conditions of an ODE

problem to solve for the integration constants. Let us illustrate this in the case of zero initial velocity $u_t(x,0) = 0$ and given deflection $u(x,0) = f(x)$.
By differentiating (1.44) we have

$$\frac{\partial u}{\partial t} = c\phi'(x+ct) - c\psi'(x-ct), \tag{1.45}$$

where primes denote derivatives with respect to the *entire* arguments $x+ct$ and $x-ct$, respectively. From (1.44), (1.45), and the initial conditions we have two equations in two unknowns:

$$u(x,0) = \phi(x) + \psi(x) = f(x) \tag{1.46}$$

$$u_t(x,0) = c\phi'(x) - c\psi'(x) = 0. \tag{1.47}$$

From (1.47) we have $\phi' = \psi'$. Hence $\phi = \psi + k$, for k a constant and from this and (1.46) we find

$$2\psi + k = f \qquad \text{or} \qquad \psi = \frac{1}{2}(f-k).$$

Then $\phi = \frac{1}{2}(f+k)$ and with these functions the solution (1.44) becomes

$$u(x,t) = \frac{1}{2}\big[f(x+ct) + f(x-ct)\big]. \tag{1.48}$$

This maybe interpreted as follows: the initial condition splits into two wavelets of half the amplitude of the original and these propagate in opposite directions at speed c. Our result shows that the initial conditions and the boundary conditions determine the solution uniquely.

Exercise: Reconsider the problem above with general initial condition $u_t(x,0) = g(x)$.

1.5 One-Dimensional Heat Flow

The heat flow in a body of homogeneous material is governed by the heat equation

$$u_t = c^2\nabla^2 u, \qquad c^2 = \frac{K}{\sigma\rho},$$

where $u(x,y,z,t)$ is the temperature in the body, K is the thermal conductivity, σ is the specific heat capacity, ρ is the density of material of the body, ∇^2 is the Laplacian of u with respect to Cartesian coordinates x, y, z, i.e. $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$. As an important application, let us consider the temperature in a long thin bar or wire of constant cross-section and homogeneous material which is oriented along the x-axis (see Figure 1.4) and is perfectly insulated laterally, so that heat flows in the x-direction only. Then u depends only on x and time t, and the heat equation becomes the so-called **one-dimensional heat equation**

$$u_t = c^2 u_{xx}. \tag{1.49}$$

While the wave equation involves the *second* partial derivative u_{tt}, the heat equation involves the *first* derivative u_t, and we shall see that the solutions of (1.49) are entirely different from those of the wave equation, although the procedure for solving (1.49) is quite similar to that in the case of the wave equation. We shall derive (1.49) and solve it for some important types of boundary and initial conditions.

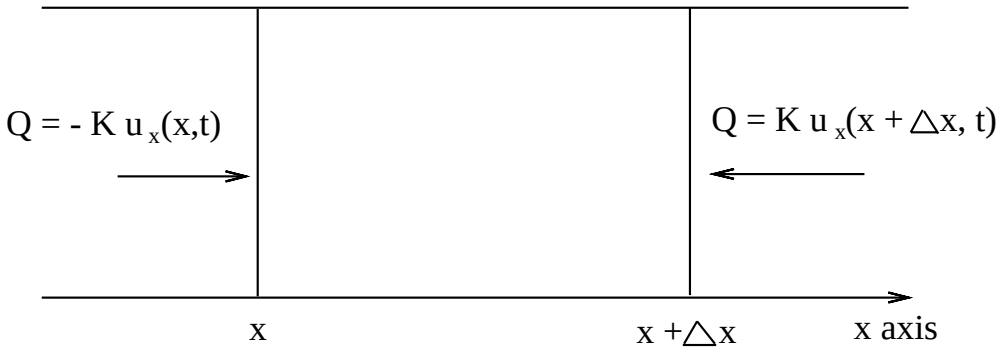

Figure 1.4: *Section of bar of material.*

1.5.1 Derivation of the heat equation

We have already derived the wave equation for vibrations in a string or membrane. We now derive the heat equation for flow in a bar of conducting material.

We assume that the *lateral sides of the bar are perfectly insulated (with some lagging material) so that there is no passage of heat through them.* We will also assume that the temperature u in the rod depends only on position x and t, and not on the lateral coordinates y and z, i.e. that the temperature in any cross-section of the rod is constant. (See Figure 1.4). This assumption is usually satisfactory when the lateral dimensions of the rod are small when compared to its length.

The differential equation governing the temperature in the bar is an expression of a fundamental physical *balance*: the rate at which heat flows into any portion of the bar is equal to rate at which heat is absorbed in that portion of the bar. The terms in the equation are called the flux (flow) term and the absorption term respectively.

We begin with Fourier law which states that the amount of heat flowing through unit cross-section area in the bar *per unit time*, called the **flux** Q, is given by:

$$Q(x,t) = -K\frac{\partial u}{\partial x}(x,t), \tag{1.50}$$

where K is the (constant) heat diffusion coefficient and depends on the material in the rod, and $u(x,t)$ is the temperature in the rod as a function of the distance along its length and time. Qualitatively the law states that if there are large differences in the temperature u along the rod i.e. if

$\frac{\partial u}{\partial x}$ is large, then heat flow occurs. This is in accordance with physical experience which testifies to the fact that heat flow tends to equalise temperatures. Also, heat flows from hot areas to cold areas (this is the origin of the negative sign in (1.50). If the temperature in the rod is everywhere the same, then $\frac{\partial u}{\partial x} = 0$ everywhere and no heat flows.

We now concentrate on an infinitesimal portion of the rod, located between the points x and $x+\Delta x$ as in Figure 1.4. In order to derive a differential equation describing the flow of heat in the bar, we will calculate the amount of heat flowing into the small element of length Δx and equate this with the increase in heat in the element arising from absorption.

At the left hand edge of the element, the *rate of heat flow per unit area*, i.e. the amount of heat flowing through a unit area in unit time to the right is given by the flux $Q(x,t)$. Similarly, the amount of heat flowing into the differential element in unit time at the right hand edge is $-Q(x+\Delta x,t)$ (Note the sign change). The net increase in heat in the differential element (per unit cross-sectional area) *in a time* Δt is thus:

$$\text{Increase in heat in element in time } \Delta t = \big(Q(x,t) - Q(x+\Delta x,t)\big)\Delta t. \tag{1.51}$$

Now the amount of heat energy per unit cross-section in the differential element at any time t is given by $\sigma\rho\Delta x u$ where σ is specific heat capacity, ρ is the density of the material and u is the average temperature in the element at time t. In fact let us take $u = u(x+\Delta x/2,t)$, i.e. u is the temperature at the centre of the element at time t. The *rate* of increase in heat in the element (i.e. the increase in heat in the element in unit time) is thus given by $\sigma\rho\Delta x u_t$ and so the net increase in heat energy in the element in time Δt is:

$$\text{Increase in heat in element in time } \Delta t = \sigma\rho\Delta x\Delta t u_t(x+\Delta x/2,t). \tag{1.52}$$

Obviously we can now equate (1.51) and (1.52) as the amount of heat flowing into the element must equal the increase in its heat energy (this is just a conservation of heat energy) and so

$$\big(Q(x,t) - Q(x+\Delta x,t)\big)\Delta t = \sigma\rho\Delta x\Delta t u_t(x+\Delta x/2,t). \tag{1.53}$$

The Δt's cancel and we now divide by both sides of Δx and take the limit as $\Delta x \to 0$ to get:

$$\lim_{\Delta x\to 0} \frac{Q(x,t) - Q(x+\Delta x,t)}{\Delta x} = \lim_{\Delta x\to 0} \sigma\rho u_t(x+\Delta x/2,t). \tag{1.54}$$

Recalling the definition of a partial derivative, (1.54) is just:

$$-Q_x(x,t) = \sigma\rho u_t(x,t). \tag{1.55}$$

But from (1.50) we have an expression for Q in terms of u so (1.50) and (1.54) become:

$$Ku_{xx}(x,t) = \sigma\rho u_t(x,t), \tag{1.56}$$

or in more familiar form of (1.49) where $c^2 = \frac{K}{\sigma\rho}$ is called the thermal diffusivity. It is a parameter depending only on the material of the bar. Its units are (length)^2/time. Typical values for different materials: silver 1.71×10^{-4} m^2 s^{-1}, brick 3.8×10^{-7} m^2 s^{-1}, water 1.44×10^{-7} m^2 s^{-1}.

1.5.2 Physical boundary conditions

Firstly, what type of PDE is the heat equation and what type of boundary conditions do we expect from the general theory earlier in the Chapter?

Several relatively simple conditions may be imposed at the end of the bar. For example, the temperature at the end may be maintained at some constant value T. This might be accomplished by placing the end of the bar in thermal contact with some reservoir of sufficient size so that any heat that may flow between the bar and the reservoir does not noticeably alter the temperature of the reservoir. At this end the mathematical (Dirichlet) boundary condition is

$$u = T. \tag{1.57}$$

Another simple boundary condition occurs if the end is insulated (i.e. the end is surrounded by a lagging material) so that no heat passes through it. Recalling the expression (1.50) (the Fourier law) for the amount of heat (per unit time) crossing any cross-section of the bar, the condition for insulation is clearly that this quantity vanishes. Thus mathematically

$$u_x = 0, \tag{1.58}$$

is a (Neumann) boundary condition at an insulated end.

A more general type of boundary condition occurs if the rate of flow of heat through an end of the bar is proportional to the temperature there. Let us consider the end $x = 0$, where the rate of heat flow from left to right is give by $-Ku_x(0,t)$; see (1.50). Hence the rate of heat flow out of the bar (from right to left) at $x = 0$ is $KAu_x(0,t)$. If this quantity is proportional to the temperature $u(0,t)$, then we obtain the (Robin) boundary condition

$$u_x(0,t) - h_1 u(0,t) = 0, \qquad t > 0, \tag{1.59}$$

where h_1 is a known non-negative constant of proportionality. Note that $h_1 = 0$ corresponds to an insulated end, while $h_1 \to \infty$ corresponds to an end held at zero temperature. (The equation then reduces to $u(0,t) = 0$).

If the heat flow is taking place at the right hand end of the bar ($x = 1$), then in a similar way we obtain the boundary condition

$$u_x(1,t) - h_2 u(1,t) = 0, \qquad t > 0, \tag{1.60}$$

where again h_2 is a known non-negative constant of proportionality.

Finally to determine completely the flow of heat in the bar it is necessary to state the temperature distribution at one fixed instant, usually taken as the initial time $t = 0$. This initial condition is of the form

$$u(x, 0) = f(x), \qquad 0 \leq x \leq 1. \tag{1.61}$$

The mathematical problem is then to determine the solution of the differential equation (1.49) subject to the boundary conditions (1.59) and (1.60) at each end, and to the initial condition (1.61) at $t = 0$.

1.5.3 Solutions to heat flow problems with homogeneous boundary conditions

Problem Formulation

We consider the case of a bar of heat conducting material of length l. The partial differential equation describing the conduction of heat through the bar is:

$$c^2 u_{xx} = u_t. \tag{1.62}$$

Let us start with the case when the ends $x = 0$ and $x = l$ of the bar are kept at temperature zero. Then the (homogeneous) *boundary conditions* are

$$u(0, t) = 0, \quad u(l, t) = 0, \qquad \text{for all } t. \tag{1.63}$$

Let $f(x)$ be the initial temperature in the bar. Then the *initial condition* is

$$u(x, 0) = f(x), \tag{1.64}$$

where $f(x)$ is a given function.

Solution to the formulated problem

First Step:
Using the method of separating variables, we first determine solutions of the equation that satisfy the boundary conditions. We start from

$$u(x, t) = F(x)G(t). \tag{1.65}$$

Differentiating and substituting this into (1.60) gives

$$F\dot{G} = c^2 F'' G,$$

where, as before, dots denote derivatives with respect to t and dashes denote derivatives with respect to x. Dividing by $c^2 FG$ (the usual trick is to divide by FG but we also divide by c^2 for convenience) we have

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F}. \tag{1.66}$$

The expression on the left depends only on t, while the right side depends only on x. As in § 4.4, we conclude that both expressions must be equal to a constant, say k. You can show that for $k \geq 0$ the only solution $u = FG$ that satisfies the boundary conditions is $u = 0$. For negative $k = -p^2$ we obtain

$$\dot{G}/c^2G = F''/F = -p^2,$$

and from this the two ordinary differential equations

$$F'' + p^2F = 0, \tag{1.67}$$

and

$$\dot{G} + c^2p^2G = 0. \tag{1.68}$$

Second Step:

The general solution of (1.67) is

$$F(x) = A\cos px + B\sin px. \tag{1.69}$$

From the boundary conditions (1.63) it follows that

$$u(0,t) = F(0)G(t); \quad u(l,t) = F(l)G(t) = 0.$$

Since $G \equiv 0$ implies $u \equiv 0$, we require that $F(0) = 0$ and $F(l) = 0$. We can conclude from (1.69) that $F(0) = A$ and so $A = 0$, therefore

$$F(l) = B\sin pl.$$

We must have $B \neq 0$, since otherwise $F \equiv 0$. Hence the condition $F(l) = 0$ leads to

$$\sin pl = 0 \qquad \text{i.e} \qquad p = n\pi/l, \quad n = 1, 2, 3....$$

We thus obtain the infinity of solutions

$$F_n(x) = B_n \sin\frac{n\pi x}{l}, \quad n = 1, 2, 3...,$$

where B_n is a completely arbitrary constant.

For the values $p = n\pi/l$ the ODE (1.68) takes the form

$$\dot{G} + \lambda_n^2 G = 0 \qquad \text{where} \quad \lambda_n = cn\pi/l.$$

The general solution is

$$G_n(t) = C_n e^{-\lambda^2 t}, \quad n = 1, 2, 3...,$$

where C_n is an arbitrary constant. Hence the functions

$$u_n(x,t) = F_n(x)G_n(t) = D_n \sin\frac{n\pi x}{l}\, e^{-\lambda_n^2 t} \quad n = 1, 2, 3..., \tag{1.70}$$

are solutions of the heat equation (1.62) satisfying (1.63). In (1.70) we have written $B_nC_n = D_n$, as B_n and C_n are arbitrary (as is D_n).

Third Step:

To find a solution which also satisfies the initial condition (1.64), we consider the series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l}\, e^{-\lambda_n^2 t}, \qquad (\lambda_n = cn\pi/l). \tag{1.71}$$

From this and the initial condition (1.64) it follows that

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l} = f(x)$$

Hence for the general solution (1.71) to satisfy (1.64), the coefficients D_n must be chosen such that $u(x,0)$ becomes a half-range expansion of $f(x)$, namely, the Fourier sine series of $f(x)$: that is

$$D_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\, dx, \qquad n = 1, 2, 3.... \tag{1.72}$$

Therefore the series (1.71) with coefficients (1.72) is the solution of the heat equation (1.62).

Example 1.5.1 *If the initial temperature in a bar of length l is given by*

$$f(x) = \begin{cases} x & \text{when } 0 < x < l/2 \\ l - x & \text{when } l/2 < x < l, \end{cases}$$

(see Figure 1.5, dotted line), with $l = \pi$ and $c = 1$, then we obtain from (1.72)

$$D_n = \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\, dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\, dx\right]. \tag{1.73}$$

Integration (both integrals in (1.73) require integration by parts) yields $D_n = 0$ when n is even and

$$D_n = \frac{4l}{n^2\pi^2}, \qquad n = 1, 5, 9, ...$$
$$D_n = -\frac{4l}{n^2\pi^2}, \qquad n = 3, 7, 11, ...$$

Hence the solution is

$$u(x,t) = \frac{4l}{\pi^2}\left\{\sin\frac{\pi x}{l}\exp\left[-\left(\frac{c\pi}{l}\right)^2 t\right] - \frac{1}{9}\sin\frac{3\pi x}{l}\exp\left[-\left(\frac{3c\pi}{l}\right)^2 t\right] + \dots\right\}.$$

In Figure 1.5 we graph this at different times (e.g. $t = 0$, 1, 2, 3) to get an instantaneous "picture" of the temperature in the bar at each of these times. Note the smoothing effect of the diffusion operator.

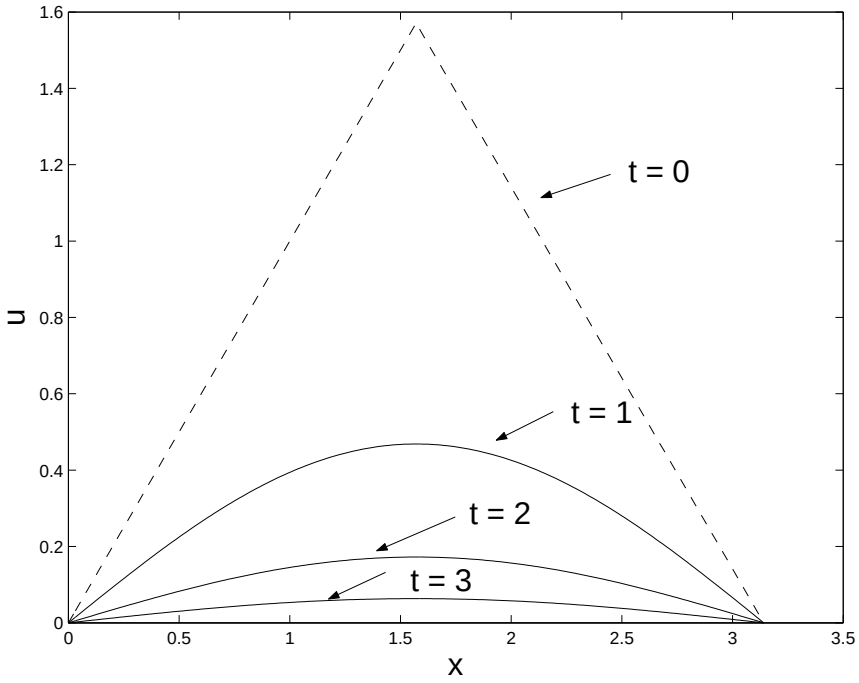

Figure 1.5: *Solution at various times.*

1.6 Laplace's Equation

One of the most important partial differential equations appearing in physics is Laplace's equation

$$\nabla^2 u = 0, \tag{1.74}$$

where, with respect to cartesian co-ordinates x, y, z in space, $\nabla^2 = u_{xx} + u_{yy} + u_{zz}$. The theory of solutions of Laplace's equation is called potential theory. Solutions of (1.74) which have continuous second order partial derivatives are called **harmonic functions**.

The two dimensional case, when u depends on x and y only, can be treated using the methods of complex analysis exploiting the fact that the real and imaginary parts of *any* analytic complex valued function $f(z) = u(x, y) + iv(x, y)$ both satisfy Laplace's equation in two dimensions. Many problems involving the two dimensional Laplace's equation reduce to exercises in complex variable theory.

1.6.1 Basic applications

We mention briefly some applications of Laplace's equation. In electrostatics the electrical force of attraction between charged particles is called Coulomb's law, which is of the same mathematical form as Newton's law of gravitation. It can be shown that the field created by a distribution of electrical charges can be described mathematically by a potential function which satisfies Laplace's equation at any point not occupied by charges. A similar result holds for the gravitational potential, i.e. the

gravitational force between two particles is given by the gradient of a scalar function (potential) which satisfies Laplace's equation.

For incompressible flow the velocity potential ϕ can be shown to satisfy $\nabla^2\phi = 0$.

Finally, in a steady state heat flow problem, the temperature u also satisfies the Laplace equation as $\partial/\partial t \equiv 0$ and the heat flow equation reduces to Laplace's equation.

1.6.2 Laplace's equation in a rectangle

We consider the following physical problem. A thin rectangular plate has its edges fixed at temperatures zero on three sides and $f(y)$ on the remaining side, as shown in Figure 1.6. Its lateral sides are then insulated and it is allowed to stand for a "long" time (but the edges are maintained at the aforementioned boundary temperatures). We wish to find the temperature distribution in the plate.

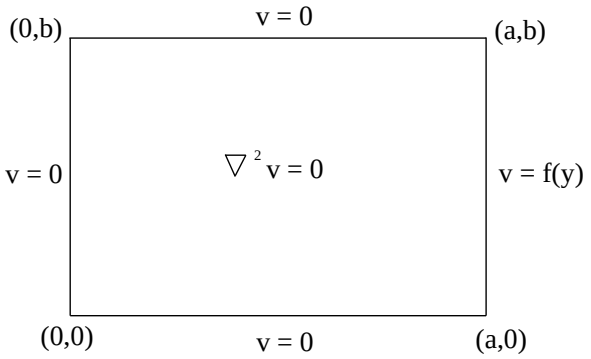

Figure 1.6: *A thin metal rectangular plate for Laplace's equation.*

Problem Formulation

The mathematical problem can be formulated as follows (PDE and boundary conditions):

$$v_{xx} + v_{yy} = 0 \quad (\nabla^2 v = 0). \tag{1.75}$$

As the equation is elliptic, we expect just one boundary condition along the boundary (in this case the conditions are Dirichlet), i.e. along each side of the rectangle. Thus we require

$$v(x,0) = 0, \qquad v(x,b) = 0, \qquad 0 \le x \le a \tag{1.76}$$

$$v(0,y) = 0, \qquad v(a,y) = f(y), \qquad 0 \le y \le b, \tag{1.77}$$

where $f(y)$ is a known function.

Solution to formulated problem

We now proceed by separation of variables: before doing so we comment on one apparent difference between the present problem and the simple hyperbolic and parabolic problems we looked at earlier in the Chapter. In these cases, we had both initial and boundary conditions and the method only worked if the boundary conditions were all homogeneous (in order to determine the eigenfunctions). The non-zero **initial conditions** were then used to solve for the superposition constants. In the present case, we have no *initial conditions* as the problem is time independent. If all our boundary conditions were homogeneous (i.e. $f(y) = 0$ in (1.77)), then the problem would have the trivial solution everywhere. In fact, the method which we are about to develop will work when we have precisely **one** *non-zero boundary condition* and *three zero conditions*. The zero or homogeneous conditions are used to solve for the superposition constants and does the same task as the initial conditions for the hyperbolic and parabolic equations.

We now seek a solution in the form:

$$v(x, y) = F(x)G(y),$$

which gives

$$G(y)F''(x) + F(x)\ddot{G}(y) = 0,$$

and so

$$F''(x) = k^2 F(x) \tag{1.78}$$

$$\ddot{G}(y) = -k^2 G(y), \tag{1.79}$$

where we have assumed the separation constant is negative. As before choosing a different sign leads to a trivial solution. In the present case, the sign depends on whether the inhomogeneous condition occurs on a $x =$constant or $y =$constant boundary. The former case applies here (see (1.77)), while if the inhomogeneous condition were on $y = b$, say, then we would have to choose a separation constant of opposite sign.

Equation (1.78) leads to the following solution:

$$F(x) = A\cosh kx + B\sinh kx, \tag{1.80}$$

and application of the first condition in (1.77) indicates $A = 0$ and we are left with

$$F(x) = B\sinh kx. \tag{1.81}$$

Equation (1.79) has solution:

$$G = C\cos ky + D\sin ky.$$

The first condition in (1.76) indicates that $C = 0$ and so

$$G = D \sin ky.$$

Then the second condition in (1.76) yields the following eigenvalue equation:

$$\sin kb = 0 \quad \Longrightarrow \quad kb = n\pi, \qquad n = 1, 2, \ldots, \infty. \tag{1.82}$$

Thus superposing over all values of n, we have:

$$v(x, y) = \sum_{n=1}^{\infty} E_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}, \tag{1.83}$$

where, as usual, we have absorbed the constants ($E_n = B_n D_n$). We now use the inhomogeneous boundary condition to solve for E_n. Thus, on $x = a$, the second condition in (1.77) gives

$$f(y) = \sum_{n=1}^{\infty} E_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b}, \tag{1.84}$$

and so by the orthogonality of the sines we have

$$E_n \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy,$$

and so

$$E_n = \frac{2}{b} \frac{\int_0^b f(y) \sin \frac{n\pi y}{b} \, dy}{\sinh \frac{n\pi a}{b}}. \tag{1.85}$$

Then the solution to the problem is complete.

Example 1.6.1 *Solve Laplace's equation on the rectangle* $0 \le x \le 3$ *and* $0 \le y \le 2$ *with*

$$f(y) = \begin{cases} y & \text{when } 0 < y < 1 \\ 2 - y & \text{when } 1 < y < 2. \end{cases}$$

We need to find the E_n *using (1.85). Now, since* $a = 3$ *and* $b = 2$ *we have*

$$\begin{aligned} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy &= \int_0^1 y \sin \frac{n\pi y}{2} \, dy + \int_1^2 (2 - y) \sin \frac{n\pi y}{2} \, dy \\ &= \frac{8}{n^2 \pi^2} \sin \frac{n\pi}{2}, \end{aligned}$$

using integration by parts. Thus

$$E_n = \frac{8 \sin \frac{n\pi}{2}}{n^2 \pi^2 \sinh \frac{3n\pi}{2}}.$$

The solution is given in (1.7a) and corresponding contour plots in (1.7b).

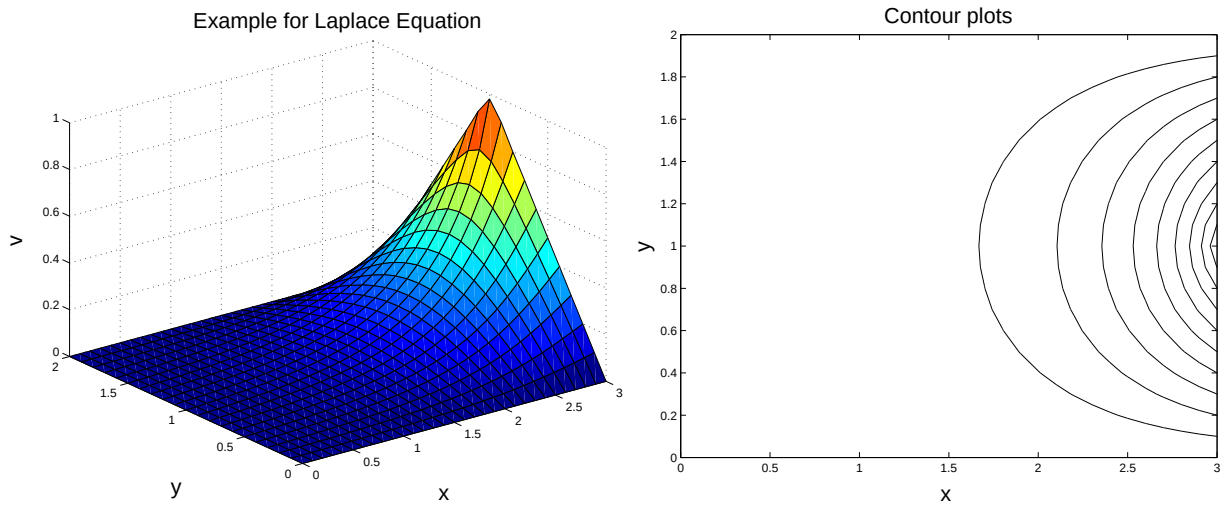

Figure 1.7: *Solution of Laplace's equation for the example (left) and corresponding contour plots (right).*

1.7 Using Integral transforms to solve PDEs

The general form of an integral transform is

$$\hat{f}(s) = \int_a^b f(x)K(s,x)\,dx, \tag{1.86}$$

where $K(s,x)$ is a *known* function of s and x and is called the kernel of the transformation. For example, the kernel of a Laplace transform would be e^{-sx}.

The effect of applying an integral transform to a PDE is to exclude temporarily a chosen independent variable and to leave for solution a PDE in one less independent variable. You will already have seen how to solve ODEs using Laplace transforms: the ODE reduces to an *algebraic* equation which is of course much easier to solve than the original ODE. Solution of this equation yields an expression for the *transform* of the dependent variable and the only remaining difficulty is in inverting to find the dependent variable itself. Generally speaking, transforming a PDE with n independent variables reduces to a PDE with $(n-1)$ independent variables (and so a PDE with two independent variables reduces to an ODE).

1.7.1 General procedure for using transforms

We must follow these steps:

1. Select the appropriate transform (depending on the equation and especially the boundary conditions).

2. Multiply the equation **and** the boundary conditions by the kernel and integrate between the appropriate limits with respect to the variable selected for exclusion.

3. In performing the integration in step 2. make use of the appropriate boundary (or initial) conditions in evaluating terms at the limits of integration.

4. Solve the resulting equations, so obtaining the *transform* of the wanted function (solution).

5. Invert to find the solution itself.

1.7.2 Definitions and summary of properties of common transforms

In the following definitions, we consider transforms of some function $f(x)$ or $f(t)$. The definitions can just as easily be extended to a function $f(x,t)$ where either the "t" variable would be unaffected and the transform would be $\hat{f}(s,t)$, or the "x" variable would be unaffected and the transform would be $\hat{f}(x,s)$.

1.7.3 Fourier transform

This is essentially a restatement of the Fourier integral theorem. Given some function $f(x)$ (satisfying certain smoothness requirements), its Fourier transform $\hat{f}(\omega)$ is defined to be

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx, \tag{1.87}$$

where $i^2 = -1$. The kernel is thus $e^{-i\omega x}$. Note that the use of ω as the variable in the kernel is purely a matter of convention. In the introduction to transforms at the beginning of § 4.7, s was used as the general transform variable. The inverse transform is now defined by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x}\, d\omega. \tag{1.88}$$

Note how in (1.87) the x dependence has been integrated out and it returns in (1.88). The notation $\mathbf{F}(f) = \hat{f}$ and $\mathbf{F}^{-1}(\hat{f}) = f$ is sometimes used.

Properties of the Fourier transform

These are stated here without proof (except for (ii)):

(i) **Linearity of the transform and its inverse**. For α, β any scalars and f, g any transformable functions:

$$\mathbf{F}(\alpha f + \beta g) = \alpha \mathbf{F}(f) + \beta \mathbf{F}(g),$$

with a similar result for the inverse transform $\mathbf{F}^{-1}$.

(ii) **Transform of the n^{th} derivative**. If $f^{(n-1)}(x)$, $f^{(n-2)}(x)$,...,$f(x) \to 0$ as $x \to \pm\infty$, then

$$\mathbf{F}\left(\frac{d^n f}{dx^n}\right) = (i\omega)^n \mathbf{F}(f) = (i\omega)^n \hat{f}(\omega).$$

Proof:

$$\begin{aligned}\mathbf{F}\left(\frac{df}{dx}\right) &= \int_{-\infty}^{\infty} \frac{df}{dx} e^{-i\omega x}\, dx \\ &= \left[f(x)e^{-i\omega x}\right]_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx \\ &= i\omega \mathbf{F}(f) = i\omega \hat{f}(\omega),\end{aligned}$$

where we have used integration by parts. Second and higher derivatives are transformed in a similar manner. Note that this his how the transform removes derivatives w.r.t. x from the equation. For example $\mathbf{F}\left(\frac{d^2 f}{dx^2}\right) = (i\omega)^2 \hat{f}(\omega)$.

(iii) **Fourier convolution**. If we define the Fourier convolution of two functions to be

$$f \star g \equiv \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi,$$

then $\mathbf{F}(f \star g) = \hat{f}(\omega)\hat{g}(\omega)$ or $f \star g = \mathbf{F}^{-1}(\hat{f}\hat{g})$. Recall that $\mathbf{F}$ and $\mathbf{F}^{-1}$ are inverses of one another so that $\mathbf{F}\mathbf{F}^{-1} = \mathbf{F}^{-1}\mathbf{F}$ is the identity transformation.

(iv) For reference purposes, we list some other less important properties:

$$\begin{aligned}&\mathbf{F}\left[(-ix)^n f(x)\right] = \hat{f}^{(n)}(\omega) \\ &x \text{ shift}: \quad \mathbf{F}^{-1}\left[e^{-ia\omega}\hat{f}(\omega)\right] = f(x-a) \\ &\omega \text{ shift}: \quad \mathbf{F}^{-1}\left[\hat{f}(\omega-a)\right] = e^{iax} f(x) \\ &\text{If} \quad h(x) = \int_{-\infty}^{\infty} f(\xi)\, d\xi \quad \text{and} \quad h \to 0 \quad \text{as} \quad x \to 0 \quad \text{then} \quad \mathbf{F}[h(x)] = \hat{f}(\omega)/i\omega.\end{aligned}$$

1.7.4 Laplace transform

The Laplace transform is defined by

$$\hat{f}(s) = \int_0^{\infty} f(t)e^{-st}\, dt, \tag{1.89}$$

where s is chosen so that the integral converges and in general can be a complex number. The inverse transformation is

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{f}(s)e^{st}\, ds. \tag{1.90}$$

Sometimes the notation $\mathbf{L}(f) = \hat{f}$ and $\mathbf{L}^{-1}(\hat{f}) = f$ is used. Note that γ and ω are real (with $\mathrm{Re}(s) > \gamma$) in the definition of the inverse transform so the transform variable $s = \gamma + i\omega$ is complex. The path of integration in (1.90) is the straight line from $\gamma - i\omega$ to $\gamma + i\omega$ in the complex s plane so inverting Laplace transforms (and indeed Fourier transforms) can be very difficult. We will generally contend ourselves with using tables of Laplace transforms to find inverse transforms or only considering special simplified situations but it should be remembered that inverting the Laplace transform of an arbitrary function is usually non-trivial and must often be undertaken numerically.

Properties of the Laplace transform

These are stated here without proof (except for (ii)):

(i) **Linearity of the transform and its inverse**. For α, β any scalars and f, g any transformable functions:

$$\mathbf{L}(\alpha f + \beta g) = \alpha \mathbf{L}(f) + \beta \mathbf{L}(g),$$

with a similar result for the inverse transform $\mathbf{L}^{-1}$.

(ii) **Transform of the n^{th} derivative**. If $f^{(n-1)}(t)$, $f^{(n-2)}(t)$,...,$f(t) \to 0$ as $t \to +\infty$, then

$$\mathbf{L}\left(\frac{d^n f}{dt^n}\right) = s\mathbf{L}\left(\frac{d^{n-1} f}{dt^{n-1}}\right) - \frac{d^{n-1} f}{dt^{n-1}}(t=0).$$

For example, for $n = 1$ and $n = 2$ we have:

$$\mathbf{L}\left(\frac{df}{dt}\right) = s\hat{f}(s) - f(0)$$
$$\mathbf{L}\left(\frac{d^2 f}{dt^2}\right) = s^2\hat{f}(s) - sf(0) - f'(0).$$

Proof:

$$\begin{aligned}\mathbf{L}\left(\frac{df}{dt}\right) &= \int_0^\infty \frac{df}{dt} e^{-st}\, dt \\ &= \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}\, dt \\ &= -f(0) + s\hat{f}(s),\end{aligned}$$

where we have used integration by parts. Second and higher derivatives are transformed in a similar manner. Note that this is how the transform removes derivatives w.r.t. t from the equation.

(iii) **Laplace convolution**. If we define the Laplace convolution of two functions to be

$$f \star g \equiv \int_0^\infty f(\tau)\, g(t-\tau)\, d\tau,$$

then $\mathbf{L}(f \star g) = \hat{f}(s)\hat{g}(s)$ or $f \star g = \mathbf{L}^{-1}(\hat{f}\hat{g})$. Recall that $\mathbf{L}$ and $\mathbf{L}^{-1}$ are inverses of one another so that $\mathbf{L}\mathbf{L}^{-1} = \mathbf{L}^{-1}\mathbf{L}$ is the identity transformation.

(iv) For reference purposes, we list some other less important properties:

$$\mathbf{L}\big[(-t)^n f(t)\big] = \hat{f}^{(n)}(s), \quad \text{e.g.} \quad \mathbf{L}\big[tf(t)\big] = -\frac{d\hat{f}}{ds}$$

$$t \text{ shift}: \quad \mathbf{L}^{-1}\big[e^{-as}\hat{f}(s)\big] = H(t-a)f(t-a)$$

$$s \text{ shift}: \quad \mathbf{L}^{-1}\big[\hat{f}(s+a)\big] = e^{-at}f(t)$$

$$\mathbf{L}^{-1}\left[\frac{\hat{f}(s)}{s}\right] = \int_0^t f(\tau)\,d\tau$$

$$\mathbf{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \hat{f}(s)\,ds$$

If $f(t)$ is periodic with period over $0 \le t < \infty$, then $\hat{f}(s) = \dfrac{1}{1-e^{-as}}\displaystyle\int_0^a f(t)e^{-st}\,dt.$

Note that H is the Heaviside step function defined as $H(t) = 0$ if $t < 0$, $H(t) = 1$ if $t > 0$.

1.7.5 General comments on when to use different transforms

Fourier and Laplace transforms can only be used on **linear** equations and are usually only useful on equations with **constant coefficients**. If $u = u(x,t)$ is governed by an equation which are functions of x but not t, then transforming w.r.t. t will reduce it to an ODE problem whose coefficients are also a function of x. If the original PDE has coefficients which are functions of t and we transform out the t variable, then the transformed problem will still be a PDE but may (in certain circumstances) be simpler than the original PDE.

If the independent variable ranges over $(0,\infty)$, the Laplace transform should be considered. If it ranges over $(-\infty,\infty)$ the Fourier transform should be more suitable.

1.7.6 Examples using different transforms

The general theory of the previous sections involved functions of a single variable $f(x)$ or $f(t)$ and transforms of these functions. The extension to functions of two or more variables is straightforward. For example, suppose that $u = u(x,t)$ and we wish to solve a PDE involving u by using integral transforms. We initially need to establish which transform we are using and **which variable we are transforming out** as there are two possible independent variables to choose from. Suppose that we decide to use a Laplace transform with respect to the t variable. Then **derivatives w.r.t. the x variables are unaffected by the transform** because the x and t variables are independent. Specifically, using (1.89):

$$\mathbf{L}\left[\frac{\partial u}{\partial x}\right] = \int_0^\infty \frac{\partial u}{\partial x}e^{-st}\,dt = \frac{\partial}{\partial x}\left\{\int_0^\infty ue^{-st}\,dt\right\} = \frac{\partial \hat{u}}{\partial x}(x,s),$$

so the $\partial/\partial x$ can be thought of as moving "outside" the transform process because the limits of the integral are independent of x. Similar results hold for higher derivatives w.r.t. x of course.

The advective equation $cu_x + u_t = 0$ where c is a constant can model wave-like phenomena and is related to the wave equation but it is of course lower order. Consider the following mathematical problem:

Example 1.7.1 *Solve $cu_x + u_t = 0$ subject to the initial conditions $u(x,0) = e^{-x}$, for $0 \leq x < \infty$, and the boundary condition $u(0,t) = e^{ct}$.*

Solution. We will use the Laplace transform in the time variable, i.e. (1.89) becomes

$$\hat{u}(x,s) = \int_0^\infty u(x,t)e^{-st}\,dt,$$

and so transforming the equation gives

$$c\frac{\partial \hat{u}}{\partial x}(x,s) + s\hat{u}(x,s) - u(x,0) = 0, \tag{1.91}$$

while transforming the boundary condition (which involves t) yields

$$\hat{u}(0,s) = \mathbf{L}\big[e^{ct}\big] = \int_0^\infty e^{-(s-c)t}\,dt = \frac{1}{s-c}. \tag{1.92}$$

The initial condition is $u(x,0) = e^{-x}$ and so (1.91) reduces to

$$\frac{\partial \hat{u}}{\partial x}(x,s) + \frac{s}{c}\hat{u}(x,s) = \frac{1}{c}e^{-x}. \tag{1.93}$$

This is just a first order linear ODE: the integrating factor is $e^{\int (s/c)\,dx} = e^{sx/c}$. Multiplying by this and rearranging the LHS gives:

$$\frac{\partial}{\partial x}\big[\hat{u}e^{sx/c}\big] = \frac{1}{c}e^{-x+sx/c}, \tag{1.94}$$

and upon integrating we obtain

$$\hat{u}e^{sx/c} = \frac{e^{-x+sx/c}}{s-c} + A(s) \qquad \text{or} \qquad \hat{u} = \frac{e^{-x}}{s-c} + A(s)e^{-sx/c}. \tag{1.95}$$

Note that the integration "constant" is in fact an arbitrary function of s written as $A(s)$ here. Referring to (1.92) we note that when $x = 0$ we require $\hat{u}(0,s) = 1/(s-c)$ and so we must set $A = 0$ in (1.95). Taking the inverse transform of both sides (recalling that $\mathbf{L}^{-1}[1/(s-c)] = e^{ct}$) we find that

$$u(x,t) = e^{-x+ct} = e^{-(x-ct)}. \tag{1.96}$$

Checking back we see that this does satisfy our governing equation and initial condition. A little reflection will indicate that this can be interpreted as a "wave" moving to the right hand side at speed c. In fact if we had taken the initial condition to be $u(x,0) = f(x)$, the solution to this problem would have been $u(x,t) = f(x-ct)$ and the initial condition propagates to the right at wave speed c.

The advective equation is obviously related to the wave equation $c^2 u_{xx} = u_{tt}$ and may be regarded as half the wave equation in some sense. Recall that D'Alembert's solution to the wave equation consisted of two waves, one moving to the right and one to the left. The advective equation generates a single wave moving to the right, while obviously its sister equation $cu_x - u_t = 0$ generates another single wave propagating to the left.

Example 1.7.2 *Solve the wave equation $c^2 u_{xx} = u_{tt}$ for a semi-infinite string by Laplace transforms given that:*

$$u(x,0) = 0 \qquad \textit{(string is initially undisturbed)}$$
$$u_t(x,0) = xe^{-x/a} \qquad \textit{(initial velocity of the string is given)}$$
$$u(0,t) = 0, \quad t \geq 0 \qquad \textit{(string is fixed at } x = 0\textit{)}$$
$$u(x,t) \to 0 \quad \textit{as} \quad x \to \infty \quad \textit{for } t \geq 0 \qquad \textit{(string is held at infinity).}$$

Solution. In this instance we take a Laplace transform with respect to the time variable t. We now need to transform the equation and its derivatives. Note that as we are transforming with respect to t the derivatives with respect to x are unaffected. For example

$$\mathbf{L}[u_x] = \mathbf{L}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\{\mathbf{L}[u]\} = \frac{\partial \hat{u}}{\partial x}(x,s),$$

and similarly

$$\mathbf{L}[u_{xx}] = \frac{\partial^2 \hat{u}}{\partial x^2}(x,s).$$

Furthermore, from § 4.7.4 we find that

$$\mathbf{L}[u_t] = s\hat{u}(x,s) - u(x,0), \qquad \mathbf{L}[u_{tt}] = s^2\hat{u}(x,s) - su(x,0) - u_t(x,0).$$

Note again how the x variable is unchanged during the transformation. The transformed wave equation now becomes

$$c^2 \frac{\partial^2 \hat{u}}{\partial x^2}(x,s) = s^2\hat{u}(x,s) - su(x,0) - u_t(x,0).$$

This equation contains no derivatives w.r.t. s. This is the essence of the idea of using transforms. The above equation is really only an ODE (sometimes called a pseudo PDE). To solve it we use ODE methods while allowing all constants of integration to be functions of the parameter s. If we can now find a solution to this ODE, we will have found $\hat{u}(x,s)$ and to find our solution $u(x,t)$ we then apply an inverse transform.

By seeking a particular integral of the form

$$\hat{u}(x,s) = \alpha x e^{-x/a} + \beta e^{-x/a},$$

we obtain the following solution to the differential equation:

$$\hat{u}(x,s) = A(s)e^{sx/c} + B(s)e^{-sx/c} - \frac{e^{-x/a}}{c^2/a^2 - s^2}\left[x + \frac{2c^2/a^2}{c^2/a^2 - s^2}\right],$$

where $A(s)$ and $B(s)$ are arbitrary and are determined from the (transformed) boundary conditions. The last two conditions when transformed are $\hat{u}(0,s) = 0$ and $\hat{u}(x,s) \to 0$ as $x \to \infty$. From these we find that $A(s) = 0$ and $B(s) = 2c^2/(a[c^2/a^2 - s^2])$.

Now $\hat{u}(x,s)$ is fully determined. Finding inverse Laplace transforms is often extremely difficult but in the present instance, we can resort to the results from tables and the s shift result in § 4.7.4. Using the results

$$\mathbf{L}[\sinh \omega t] = \frac{\omega}{s^2 - \omega^2}, \qquad \mathbf{L}[\cosh \omega t] = \frac{s}{s^2 - \omega^2}, \qquad \mathbf{L}\left[\frac{\omega t \cosh \omega t - \sinh \omega t}{2\omega^3}\right] = \frac{1}{(s^2 - \omega^2)^2},$$

we obtain

$$\begin{aligned} u(x,t) = {} & \frac{a}{c}\big[(ct - x)\cosh\{(ct - x)/a\}H(ct - x) - cte^{-t/a}\cosh\{ct/a\}\big] \\ & + \frac{a}{c}\big[(x + a)e^{-x/a}\sinh\{ct/a\} - a\sinh\{(ct - x)/a\}H(ct - x)\big], \end{aligned}$$

where H is the Heaviside step function.

Example 1.7.3 *Consider a semi-infinite thin plate of material defined by $-\infty < x < \infty$, $0 \le y < \infty$. Assume the boundary of the plate $y = 0$ is held at a temperature $u(x,0) = f(x)$, and that all other edges have their temperatures held at 0. Formulate a boundary value problem describing the steady state heat flow problem in the plate and solve it using Fourier transforms.*

Solution. The governing equation is

$$u_{xx} + u_{yy} = 0, \tag{1.97}$$

with boundary conditions

$$u(x,0) = f(x), \tag{1.98}$$

$$u(x,\infty) = u(-\infty,y) = u(\infty,y) = 0. \tag{1.99}$$

Using the definitions of § 4.7.3, we transform out the x variable arriving at

$$\frac{\partial^2 \hat{u}}{\partial y^2}(\omega,y) - \omega^2 \hat{u}(\omega,y) = 0, \tag{1.100}$$

where we have assumed that $u_x \to 0$ as $x \to \pm\infty$. Once again (1.100) is an ODE and has solution

$$\hat{u}(\omega,y) = A(\omega)e^{-\omega y} + B(\omega)e^{\omega y}, \tag{1.101}$$

where we again must allow the arbitrary constants to be functions of ω.

Transforming the first condition in (1.99) we find $\hat{u}(\omega, \infty) = 0$ and so in (1.101) we must set $B(\omega) = 0$. Accordingly we have

$$\hat{u}(\omega, y) = A(\omega)e^{-\omega y}, \tag{1.102}$$

where $A(\omega)$ is yet to be determined. Transforming (1.98) gives

$$\hat{u}(\omega, 0) = \mathbf{F}[f(x)] = \hat{f}(\omega), \tag{1.103}$$

and from (1.102) we have $A(\omega) = \mathbf{F}[f(x)]$. As $f(x)$ is known, this determines $A(\omega)$ in principle, provided we can determine the Fourier transform of $f(x)$. By definition this is

$$\mathbf{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx, \tag{1.104}$$

and so we can write (1.102) as

$$\hat{u}(\omega, y) = e^{-\omega y} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx. \tag{1.105}$$

To find the solution $u(x, y)$ we now use the inverse transform (1.88) and obtain

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega y} e^{i\omega x} \left\{ \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx \right\} d\omega. \tag{1.106}$$

In principal this integral can be evaluated (if necessary numerically) although for particular functions $f(x)$ it may be possible to obtain an analytical solution.

Chapter 2

Numerical Methods for PDEs - Finite Difference Methods

The problem we have dealt with up to this point have been fairly easy in that the equations were linear and the domains were regular (e.g. squares, rectangles, circles etc). For more complicated problems, where the governing equation is non-linear and not susceptible to an analytic approach or the domain is irregular, it becomes necessary to use approximation or numerical techniques. We examine numerical techniques applied to *easy* (linear) PDE problems here in order to illustrate the basic ideas though the problems which we look at could be solved analytically.
The same basic considerations hold when dealing with **algebraic** equations. If an algebraic equation is **linear**, then it can be solved analytically. For example, the solution to the algebraic equation

$$4x = 3x + 5,$$

is simple to find mainly because the equation is linear in the unknown x. If the equation is non-linear in the unknown x, we cannot in general solve it except for certain well known special cases (e.g. quadratic equations like $2x^2 + 3x = 1$). However, an equation with a more general nonlinear function of x cannot usually be solved in closed form. For example, consider

$$\sin x + e^x + x^2 = 1.$$

This is highly nonlinear and cannot be solved analytically. There are however a number of ways of solving the equation **numerically**. The basic idea behind solving such equations numerically is to write the function in the form $f(x) = 0$, graph the function $f(x)$ as a function of x, and from the graph find where $f(x) = 0$, i.e. where the graph crosses the x-axis. Note that such a technique would also work for the linear equation above but it is of course not necessary in such situations. Also note that **linear** equations generally have **unique** solutions while nonlinear problems may have more than one solution (such as $x^2 = 1$). When dealing with numerical solutions of nonlinear

PDEs, the same basic considerations hold. In principle there are several different types of numerical techniques available for the solution of PDEs, e.g.

- Finite difference methods;
- Finite element methods;
- Boundary element methods.

We will restrict attention to finite difference methods. Note that all the methods follow the same basic principle. Instead of trying to solve the differential equation at every point in a domain, we try to solve it approximately at a discrete number of points, and the final solution will consist of a table of values of the unknown at a number of **nodal** points in the domain. For example, suppose we wish to solve the heat equation for heat flow in a bar of length l. Our numerical final solution, unlike an analytical solution, will not give us the solution at every point on the bar but merely at a discrete number of points along its length (e.g. at $x = 0, l/100, 2l/100, 3l/100, ..., l$) and we will not have solutions at every point in time but at a discrete number of time intervals (e.g. $t = 0, 0.1, 0.2, 0.3, ...$ etc.). If we require a solution at a particular point which does not coincide with one of these points, we find it by interpolation.

The numerical solution of PDEs is similar in philosophy to numerical solution of ODEs. The basic idea of the method is to construct approximations to derivatives in terms of the required function at discrete points. One common approximation for a derivative is the **central difference**:

$$\frac{df}{dx}(x = a) \cong \frac{f(a + h) - f(a - h)}{2h}. \tag{2.1}$$

Of course this is only an approximation but we expect the approximation to become better as $h \to 0$. For the second derivative we have the central difference approximation:

$$\frac{d^2f}{dx^2}(x = a) \cong \frac{f(a + h) - 2f(a) + f(a - h)}{h^2}. \tag{2.2}$$

Note that **backward** and **forward** difference approximations are also possible. For example, the backward difference approximation for df/dx is:

$$\frac{df}{dx}(x = a) \cong \frac{f(a) - f(a - h)}{h}, \tag{2.3}$$

and the forward difference approximation is

$$\frac{df}{dx}(x = a) \cong \frac{f(a + h) - f(a)}{h}, \tag{2.4}$$

While it might be expected that each of these approximations would give equivalent results when applied to the same problem, this is not the case and certain problems are more easily solved using central differences than backward (or forward) differences and vice versa. In general central differences are the most commonly applied. However, the type of numerical scheme used is strongly

dependent on the type of PDE (elliptic, parabolic or hyperbolic) and different methods have been developed for each type. We note that the justification for these approximations and the computation of the errors involved depends on the Taylor expansions of the functions. For example, using Taylor series

$$f(a+h) = f(a) + h\frac{df}{dx}(a) + \mathcal{O}(h^2)$$
$$f(a-h) = f(a) - h\frac{df}{dx}(a) + \mathcal{O}(h^2),$$

where $\mathcal{O}(h^2)$ means terms which behave essentially like constant times h^2 and can be expected to be small if h is chosen to be small enough. Subtracting the second equation from the first we find:

$$\frac{df}{dx}(x=a) \cong \frac{f(a+h) - f(a-h)}{2h} + \mathcal{O}(h^2), \tag{2.5}$$

so the error in this approximation is $\mathcal{O}(h^2)$. [Note that the $\mathcal{O}(h^2)$ do not cancel in the subtraction!]

Exercise: Show that the error in the backward difference for df/dx is $\mathcal{O}(h)$. This indicates that a central difference is "superior" in some sense to a backward (or forward) difference.

When dealing with partial derivatives, the approximations are the same except for the fact that there is a partial derivative with respect to each variable. For example, if $u = u(x,t)$ then an approximation for $\partial u/\partial x$ at $x = a$ is

$$\frac{\partial u}{\partial x}(x=a,t) \cong \frac{u(a+h,t) - u(a-h,t)}{2h} + \mathcal{O}(h^2). \tag{2.6}$$

2.1 Numerical solutions to the wave equation (hyperbolic)

Figure 2.1 illustrates a mesh of points, or **nodes**, with spacing Δx in the x direction and Δt in the t direction. Each node is specified by a pair of integers (i,j) so that the coordinates of the nodal points take the form:

$$x_i = x_{i-1} + \Delta x, \qquad t_j = t_{j-1} + \Delta t,$$

and then we use a forward difference at the point A and a backward difference at the point B:

$$\left(\frac{\partial u}{\partial x}\right)_A \cong \frac{u_{i,j} - u_{i-1,j}}{\Delta x}, \qquad \left(\frac{\partial u}{\partial x}\right)_B \cong \frac{u_{i+1,j} - u_{i,j}}{\Delta x},$$

i.e. a forward difference at A and a backward difference at B. Note that $u_{i,j} = u(x_i, t_j)$. The second derivative at P therefore has the (central difference) numerical form:

$$\frac{\partial^2 u}{\partial x^2} = \frac{(\partial u/\partial x)_B - (\partial u/\partial x)_A}{\Delta x} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}. \tag{2.7}$$

Similarly,

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2}. \tag{2.8}$$

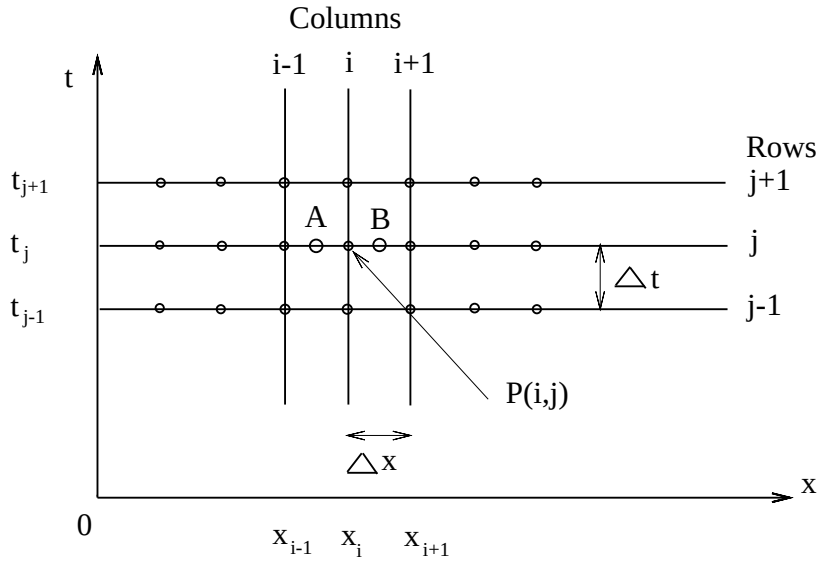

Figure 2.1: *Mesh points for a numerical solution of the wave equation.*

Thus the wave equation $u_{tt} = c^2 u_{xx}$ becomes

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} = c^2 \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2},$$

which can be re-arranged as

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \lambda^2 \big[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\big], \tag{2.9}$$

where $\lambda = c\Delta t/\Delta x$. Equation (2.9) is a **finite difference representation** of the wave equation, and provided that u is known on rows $j-1$ and j then $u_{i,j+1}$ can be computed on row $j+1$ from (2.9) and thus the solution continued. On the zeroth row the boundary conditions $u(x,0) = f(x)$ and $\partial u(x,0)/\partial t = g(x)$ are known, so that $f_i \equiv u_{i,0}$ and $g_i \equiv \partial u/\partial t\big|_{i,0}$ are also known at each node on this row, and these are used to start the process off. Using a centred difference we see that

$$g_i = \left.\frac{\partial u}{\partial t}\right|_{(i,0)} = \frac{u_{i,1} - u_{i,-1}}{2\Delta t}. \tag{2.10}$$

Now (2.9) with $j = 0$ becomes

$$u_{i,1} = 2u_{i,0} - u_{i,-1} + \lambda^2 \big[u_{i+1,0} - 2u_{i,0} + u_{i-1,0}\big]. \tag{2.11}$$

Since $u_{i,0} = f_i$ and $u_{i,-1} = u_{i,1} - 2\Delta t g_i$, (2.11) now takes the form

$$u_{i,1} = (1-\lambda^2) f_i + \frac{\lambda^2}{2}\big(f_{i+1} + f_{i-1}\big) + \Delta t g_i. \tag{2.12}$$

Thus the basic strategy is to **compute row zero from $u_{i,0} = f_i$, evaluate row one from (2.12), and then to march forward for general row j by (2.9)**.

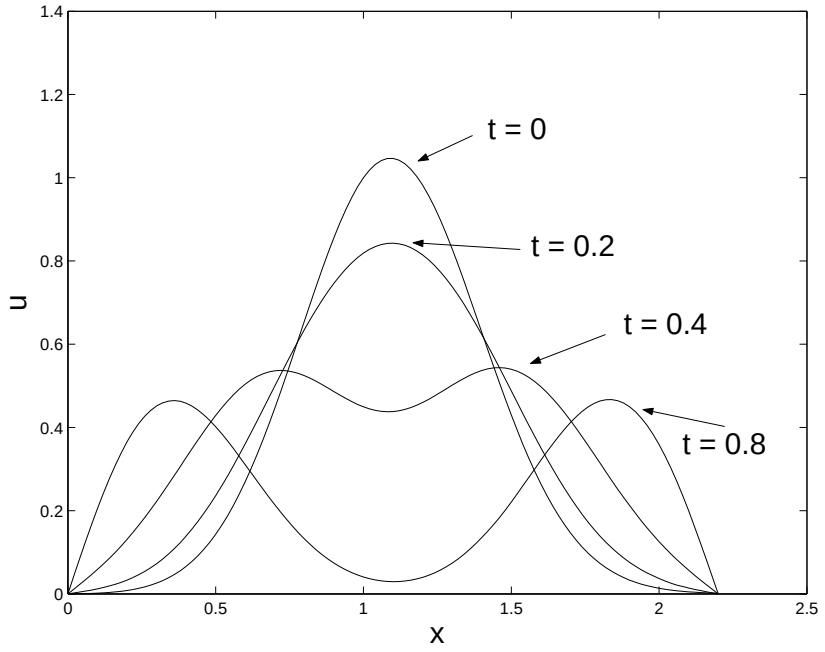

Figure 2.2: *Solution of wave equation example (with* $u(x,0) = x\exp\left[-5(x-1)^2\right]$*, with* $\Delta x = 0.02$*,* $\lambda = 0.5$ *for successive values of* t*.*

Example: Solve the wave equation $u_{tt} = c^2 u_{xx}$, with $c = 1$, for a semi-infinite string ($0 \le x < \infty$), given the initial conditions

(a) $u(x,0) = x\exp\left[-5(x-1)^2\right]$ $(x \ge 0)$ (string has a known initial displacement);

(b) $\partial u(x,0)/\partial t = 0$ $(x \ge 0)$ (string is initially at rest);

(c) $u(0,t) = 0$ $(t \ge 0)$ (string is fixed at the point $x = 0$).

Solution. Since $g_i = 0$ in (2.12), only the one parameter λ needs to be specified. Figure 2.2 shows the solution of u for various t with $\lambda = 0.5$ and $\Delta x = 0.02$. It can be seen that the solution splits into two waves, one moving in the $+x$ direction and the other in the $-x$ direction. At a given time $t = 0.8$, the u values are presented in Table 2.1 for various λ, with $\Delta x = 0.2$. We see that for $\lambda < 1$ the solution is reasonably consistent, and we have errors of a few per cent. However, for $\lambda = 4/3$ the solution looks very suspect. It should be noted that the same using $\Delta x = 0.02$ gives much more accurate results (as expected because the mesh is much smaller) but the solution blows up for $\lambda > 1$.

We can attempt an explanation for the apparent divergence of the solution in the example using Figure 2.3. The characteristics through the points $(x_{i-1}, 0)$ and $(x_{i+1}, 0)$ are:

$$x_{i-1} = x - ct, \qquad x_{i+1} = x + ct,$$

x	0	0.2	0.4	0.6	0.8	1.0
u $(\lambda = 0.5)$	0	0.3487	0.4665	0.3318	0.1340	0.0272
u $(\lambda = 0.8)$	0	0.3568	0.4634	0.3208	0.1321	0.0346
u $(\lambda = 1)$	0	0.3652	0.4682	0.3105	0.1322	0.0408
u $(\lambda = 4/3)$	0	0.4173	0.4033	0.2794	0.1994	0.0039

Table 2.1: *Table of values of u for a numerical solution for the example with* $\Delta x = 0.2$ *and* $t = 0.8$.

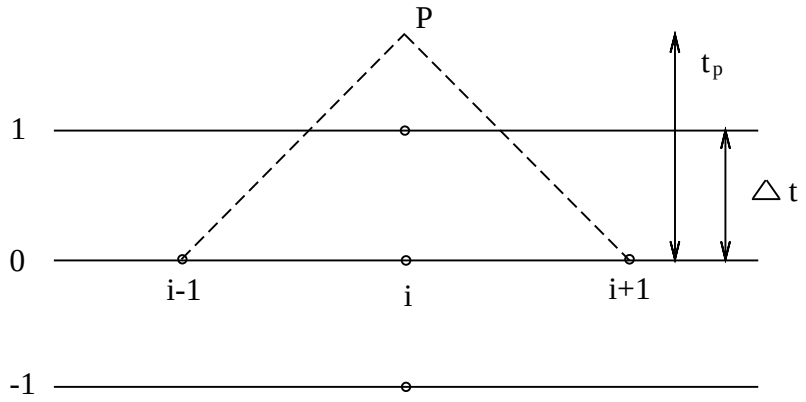

Figure 2.3: *The first rows of mesh points in the numerical solution of the wave equation.*

which can be solved to give, at the point P,

$$ct_P = \frac{1}{2}(x_{i+1} - x_{i-1}) = \Delta x.$$

Recalling the work done on characteristics, we should require the new point to be *inside* the domain of dependence defined by the interval (x_{i-1}, x_{i+1}). Hence we require

$$t_P \geq \Delta t,$$

so

$$\frac{c\Delta t}{\Delta x} \leq 1.$$

Indeed, a careful analysis, found in many specialist numerical analysis books, shows that this is precisely the condition for convergence of the method. The stringent condition on the timestep Δt has always been considered to be a limitation of so-called **explicit methods** of the type described here, but such methods have the great merit of being very simple to program. (Note that an **explicit** technique for solving two variables u and v say, is one which gives rise to equations where u and v are explicitly given in terms of wholly **known** quantities. An **implicit** solution technique might give u and v in terms of known quantities **and** each other thus requiring some further work in order to obtain values for u and v. As another illustration the single equation $u = \sin x$ defines

u **explicitly** as a function of x but the equation $u = \sin(x - u)$ defines u **implicitly** as a function of x). As computers get faster, the very short timestep required for explicit methods is becoming less of a problem, and vector or array processors allow nodes to be dealt with simultaneously, thus making such methods even more competitive.

There are, however, clear advantages in the stability of calculations if an **implicit method** is used. In Figure 2.1 the approximation to u_{xx} may be formed by the average of the approximations from rows $j+1$ to $j-1$. Thus

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{c^2\Delta t^2} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}}{2\Delta x^2}.$$

Assuming that u is known on rows j and $j-1$, we can rearrange the equation into the convenient form

$$\begin{aligned} &-\lambda^2 u_{i+1,j+1} + 2(1+\lambda^2)u_{i,j+1} - \lambda^2 u_{i-1,j+1} \\ &= 4u_{i,j} + \lambda^2 u_{i+1,j-1} - 2(1+\lambda^2)u_{i,j-1} + \lambda^2 u_{i-1,j-1}. \end{aligned} \tag{2.13}$$

The right-hand side of (2.13) is known, since it depends only on rows j and $j-1$. The unknowns on row $j+1$ appear on the left-hand side. The equations (essentially of the form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a matrix and $\mathbf{x}$, $\mathbf{b}$ are column vectors, with $\mathbf{b}$ being known and $\mathbf{x}$ the unknowns which we are seeking) can now be solved simultaneously using Gaussian elimination or algorithms for tridiagonal systems. It can be shown that the method will proceed satisfactorily for any λ, so that the timestep is unrestricted. The evaluation of rows 0 and 1 is the same as for the explicit method, so this can reduce the accuracy, and clearly the algorithm needs a finite x region to allow the matrix inversion.

2.2 Numerical solutions to the diffusion equation (parabolic)

As for the wave equation, except for the most straightforward problems, we must resort to numerical solutions of the heat-conduction equation. Even when analytical solutions are known, they are not always easy to evaluate because of convergence difficulties near to singularities. They are, of course, crucial in testing the accuracy and efficiency of numerical methods.

We can write the heat-conduction equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{2.14}$$

in the usual *finite-difference* form, using the notation of Figure 2.4.

We assume that we know the solution up to timestep j and we wish to calculate the solution at time step $j+1$. In § 5.1 we showed how to approximate the second derivative as

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}.$$

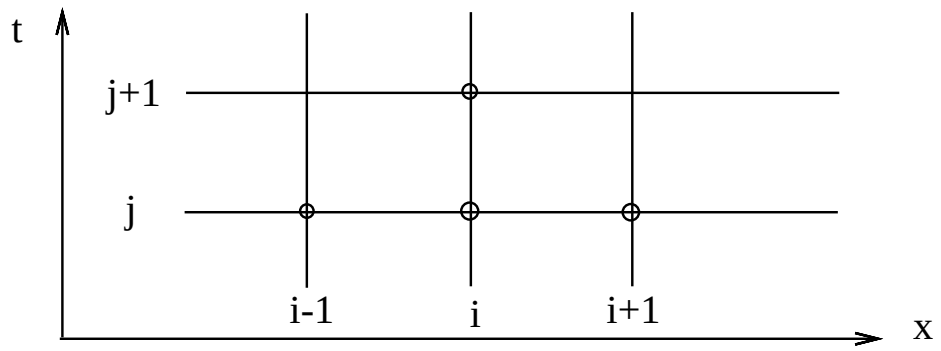

Figure 2.4: *Mesh for marching forwards in the numerical solution of the heat conduction equation.*

To obtain the time derivative, we use the approximation between rows j and $j+1$:

$$\frac{\partial u}{\partial t} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}.$$

Putting these into (2.14) gives

$$\frac{u_{i,j+1} - u_{i,j}}{c^2 \Delta t} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2},$$

or, on rearranging,

$$u_{i,j+1} = \lambda u_{i-1,j} + (1 - 2\lambda) u_{i,j} + \lambda u_{i+1,j}, \tag{2.15}$$

where $\lambda = c^2 \Delta t / \Delta x^2$. Equation (2.15) gives a finite-difference representation of (2.14), and provided that all the values are known on row j, we can then compute u on row $j+1$ from the simple **explicit formula** (2.15). This formula can only be used on the *inner* mesh points but the boundary conditions give the values $u_{0,j}$ and $u_{I,j}$ (where I is the size of the grid and so $x_I = 1$ for a domain $0 \leq x \leq 1$). We will see this now in an example.

Example: Solve the heat conduction equation $u_t = c^2 u_{xx}$, with $c = 1$, for a bar of conducting material of length 1 subject to the boundary conditions:

(a) $u(x, 0) = x^2$ $(0 \leq x \leq 1)$ (a known initial temperature distribution);

(b) $\partial u(0, t)/\partial x = 0$ $(t \geq 0)$ (the end $x = 0$ is insulated so no heat flow occurs through it);

(c) $u(1, t) = 1$ $(t \geq 0)$ (the end $x = 1$ is held fixed at temperature $u = 1$).

Solution. The condition (a) gives $u_{i,0} = x_i^2$ and this holds along the x-axis (as shown in Figure 2.5). If we choose $\Delta x = 0.2$ then the values of $u_{i,0}$ are 0^2, 0.2^2, 0.4^2, 0.6^2, 0.8^2, 1^2 (for $i = 0, 1, ..., 5$ and so here $I = 5$). The condition (c) immediately gives $u_{5,j} = 1$ and we therefore only need to interpret condition (b). We can use a central difference to deduce that

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} \equiv \frac{u_{-1,j} - u_{1,j}}{2\Delta x} = 0 \qquad \Longrightarrow \qquad u_{-1,j} = u_{1,j}. \tag{2.16}$$

We are essentially adding a fictitious line of nodes at $i = -1$. The point $u_{-1,j}$ must be eliminated in a similar way to that done for the wave equation (where we eliminated $u_{i,-1}$ using (2.10)). We use (2.15) with $i = 0$ to obtain

$$u_{0,j+1} = \lambda u_{-1,j} + (1 - 2\lambda)u_{0,j} + \lambda u_{1,j} = (1 - 2\lambda)u_{0,j} + 2\lambda u_{1,j}, \tag{2.17}$$

since $u_{-1,j} = u_{1,j}$. Then we can find all the points along the left boundary using (2.17), since the right hand side involves known values (they are all at j rather than $j + 1$). From Figure 2.5 it is clear that we have all the information required along the $x = 0$ and $x = 1$ axis, as well as along the $t = 0$ axis. We can then use (2.15) to solve for the interior points up until the appropriate j value to give us the required time. In Table 2.2 we show the results at $t = 1$ for various λ and fixed $\Delta x = 0.2$. The number of j points used is determined from Δt, which we find from rearranging the expression $\lambda = c^2\Delta t/\Delta x^2$. In Figure 2.6 we plot the solution u with $\lambda = 0.2$.

Comparing this example with the numerical solution of the wave equation from § 5.1, we observe similar behaviour for the explicit scheme, namely that the method will only converge for small enough timesteps or λ. From (2.15) it may be noted that the middle term changes sign at $\lambda = 0.5$, and above this value we might anticipate difficulties (as shown in Table 2.2). Indeed, some straightforward numerical analysis shows that convergence is certain for $\lambda < 0.5$. It is sufficient here to note that λ must not be too large.

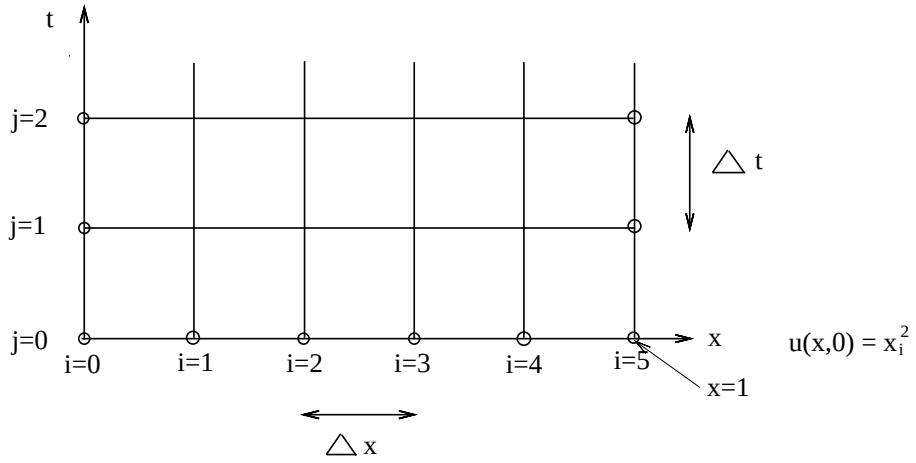

Figure 2.5: *Mesh for the example of the numerical solution of the heat conduction equation.*

To avoid the limitation on λ, we can again look at an **implicit** formulation of the numerical equations. Returning to Figure 2.4, the idea is to approximate the x derivative by an average of row j and row $j + 1$. This leads to (instead of (2.15))

$$-\lambda u_{i-1,j+1} + 2(1 + \lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1 - \lambda)u_{i,j} + \lambda u_{i+1,j}, \tag{2.18}$$

and is called the **Crank-Nicolson method**. We have the solution on row j, so the right hand

x	0	0.2	0.4	0.6	0.8	1.0
u $(\lambda = 0.2)$	0.9128	0.9171	0.9295	0.9488	0.9731	1
u $(\lambda = 0.5)$	0.9160	0.9202	0.9320	0.9507	0.9740	1
u $(\lambda = 0.625)$	-1.3568×10^3	1.2922×10^3	-1.0975×10^3	5.1857×10^2	4.1857×10^2	1

Table 2.2: *Table of values of u for a numerical solution for the heat conduction example.*

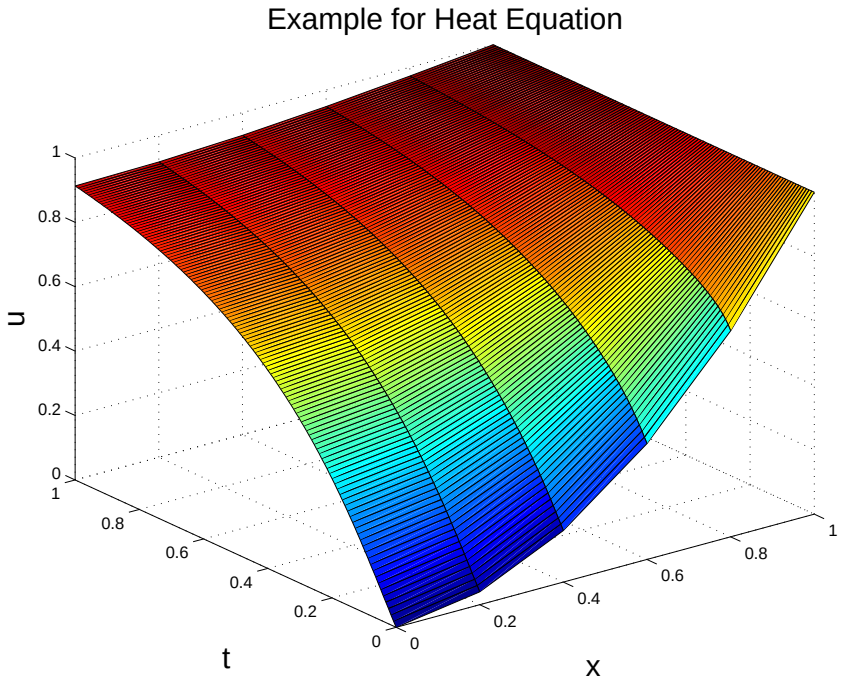

Figure 2.6: *Solution of heat equation example, with* $\Delta x = 0.2$, $\lambda = 0.2$ *and* $t = 1$.

side of (2.18) is known, and the row $j + 1$ have to be solved for simultaneously. Again this can be written in matrix form, as discussed after equation (2.13) in the previous section.

2.3 Numerical solution of elliptic problems

Of the three classical partial differential equations, the Laplace equation proves to be the most difficult to solve. The other two have a natural time variable in them, and it is possible, with a little care, to march forward either by a simple explicit method or by an implicit procedure. In the case of the Laplace equation (elliptic), information is given around the whole of the boundary of the solution region, so the field variables at *all* mesh points must be solved simultaneously. This in turn leads to a solution of the algebraic problem by matrix inversion.

The usual numerical approximation for the partial derivatives, discussed in § 5.1, are employed, so

that the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{2.19}$$

at a typical point, illustrated in Figure 2.7, becomes

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0.$$

For the case $\Delta x = \Delta y$, rearranging gives

$$4u_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}. \tag{2.20}$$

This is the typical five-point stencil and it should be noted that the middle value $u_{i,j}$ is the average of its four neighbours. We now examine how (2.20) can be implemented.

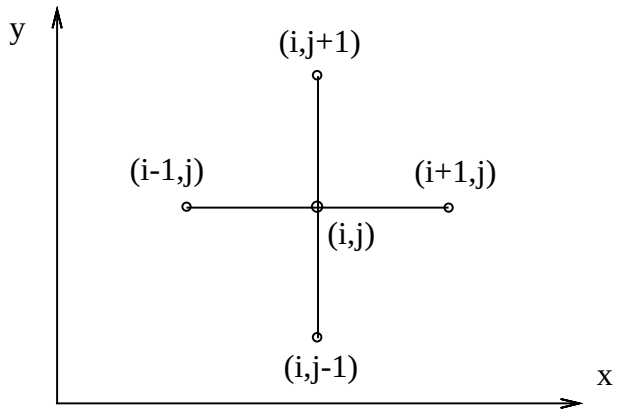

Figure 2.7: *Five-point computational model for the Laplace equation.*

Example: Solve the Laplace equation (2.19) in the square region $0 \leq x \leq 1$, $0 \leq y \leq 1$, with $\Delta x = \Delta y = 1/3$ and the following boundary conditions:

$$u(0, y) = 0, \qquad u(1, y) = 1, \qquad u(x, 0) = 0, \qquad u(x, 1) = 0.$$

Solution. For a first solution we take the simplest mesh, illustrated in Figure 2.8, which contain only four interior points labelled u_1, u_2, u_3, u_4. The four equations obtained from (2.20) are

$$4u_1 = 0 + 0 + u_2 + u_4$$

$$4u_2 = 0 + 1 + u_3 + u_1$$

$$4u_3 = 1 + 0 + u_4 + u_2$$

$$4u_4 = 0 + 0 + u_1 + u_3,$$

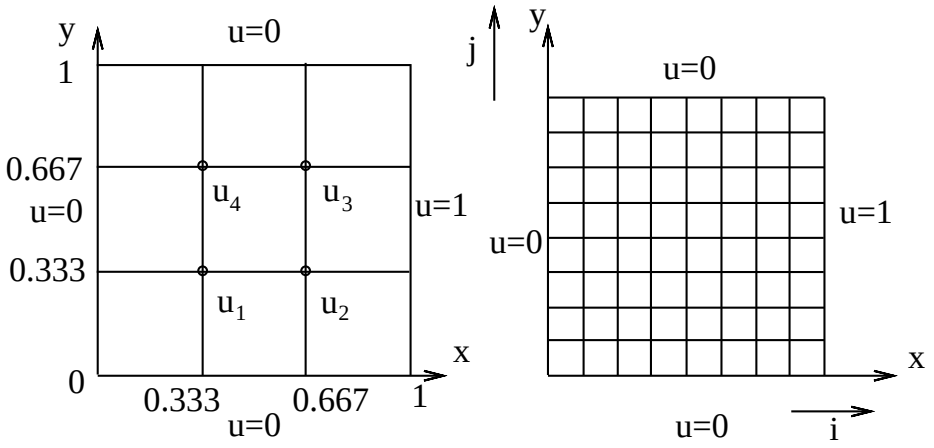

Figure 2.8: *Meshes for the solution of the Laplace equation in this example: left is a simple mesh containing 4 interior points; right is a larger mesh with 49 interior points.*

which in turn can be written in matrix form ($\mathbf{Ax} = \mathbf{b}$) as

$$\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

This has the solution $u_1 = 0.125$, $u_2 = 0.375$, $u_3 = 0.375$, $u_4 = 0.125$ which could be obtained using Matrix inversion or Cramer's rule. Computationally both of these methods are not effective for large matrices and in practice some kind of elimination is used (e.g. Gaussian reduction or Successive Over Reduction (SOR)). A larger mesh obtained by dividing the sides up into eight equal parts is indicated in Figure 2.8. This generates 49 linear equations in 49 unknowns which can be solved by any convenient matrix inverter.

We have only considered u to be given on the boundary, and it is essential to know how to deal with derivative boundary conditions where $\partial u/\partial n$ is given, since these are very common. Let us consider a typical example:

$$\frac{\partial u}{\partial x} = g(y), \quad \text{on} \quad x = 0.$$

We then insert a fictitious line of nodes, as discussed for the example in the previous section (after equation (2.16)). Approximately, the boundary condition gives

$$\frac{u_{1,j} - u_{-1,j}}{2\Delta x} = g_j,$$

so that

$$u_{-1,j} = u_{1,j} - 2\Delta x g_j. \tag{2.21}$$

Equation (2.20) is now solved for $i = 0$ as well as $i > 0$, but at the end of a sweep $u_{-1,j}$ will be updated via (2.21).

Example: Solve the Laplace equation (2.19) in the square region $0 \leq x \leq 1$, $0 \leq y \leq 1$, with mesh spacing $\Delta x = \Delta y = 1/4$ and the following boundary conditions:

$$\frac{\partial u}{\partial x}(0, y) = 1, \qquad u(1, y) = y^2, \qquad u(x, 0) = 0, \qquad u(x, 1) = x.$$

[The first condition indicates a heat supply along this boundary and the last three conditions correspond to the temperature being given along these three sides].

Solution. The region is illustrated in Figure 2.9. Since we have added the fictitious boundary $i = -1$ there are 12 unknowns. Equation (2.21) just gives $u_{-1,j} = u_{1,j} - 1/2$ for each j. From (2.20), we can write the equations as

$$\begin{aligned} 4u_{0,1} &= u_{-1,1} + u_{1,1} + 0 + u_{0,2} = -1/2 + 2u_{1,1} + u_{0,2} \\ 4u_{1,1} &= u_{0,1} + u_{2,1} + 0 + u_{1,2} \\ 4u_{2,1} &= u_{1,1} + u_{3,1} + 0 + u_{2,2}, \end{aligned}$$

and so on, and hence obtain 12 equations in the 12 unknowns. These can be solved by any convenient method to give a solution as tabulated in Table 2.3.

	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$j = 4$	0.0000	0.2500	0.5000	0.7500	1.0000
$j = 3$	-0.1190	0.1114	0.3067	0.4644	0.5625
$j = 2$	-0.1987	0.0077	0.1511	0.2384	0.2500
$j = 1$	-0.1912	-0.0330	0.0516	0.0881	0.0625
$j = 0$	0.0000	0.0000	0.0000	0.0000	0.0000

Table 2.3: *Data from the solution of the example using a step length of 0.25 in each direction.*

Finally, note that if we were to have an *inhomogeneous* Laplace equation, namely the Poisson equation

$$\nabla^2 u \equiv u_{xx} + u_{yy} = f, \tag{2.22}$$

then the finite difference approximation would become, instead of (2.20)

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j} \tag{2.23}$$

where $f_{i,j} \equiv f(x_i, y_j)$ which is known.

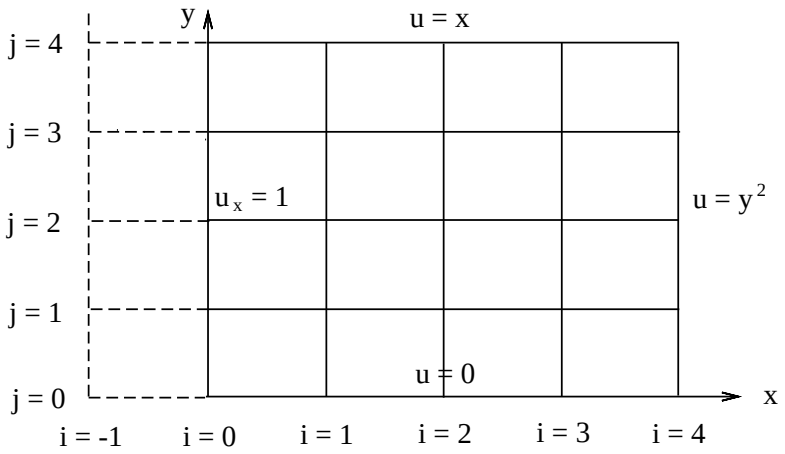

Figure 2.9: *Mesh used in the example with u_x specified on the $x = 0$ boundary.*