Non-Interacting Control

Consider a multi input - multi output (MIMO) system: generally in such a system, each input will affect many of the outputs and conversely each output will be influenced by more than one input. This can lead to such systems being difficult to control, particularly by humans e.g. a pilot attempting to control a modern high performance plane.
Consider the case of a **square** MIMO system, i.e. one in which the number of inputs equals the number of outputs ($= p$, say) If we can design a pre-compensator which decouples the interconnections so that the i-th input controls the i-th output and only that output, then we will have gone a long way to simplifying the control problem. *Falb* & *Wolovich* discovered conditions for decoupling LTI systems by linear state feedback: in fact their strategy gives a very specific structure to these decoupled systems, which makes them very amenable to subsequent control design.

1 Decoupling Indices

The ideas in this chapter are presented firstly for continuous-time LTI systems. They equally well hold, mutatis mutandis, for discrete-time LTI systems.
For a system with p inputs and p outputs

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad (1)$$

$$y_i = C_i\mathbf{x} \qquad i = 1, 2, \ldots, p \quad (2)$$

where C_i be the i-th row of output matrix C corresponding to the i-th output y_i; then the decoupling index d_i for this output is defined as the **smallest nonnegative integer** such that $C_iA^{d_i}B \neq \mathbf{0}$. By repeatedly differentiating Eq (2), substituting from Eq (1) and using the definition of the decoupling index, we generate the sequence

$$\begin{aligned}
\frac{d}{dt}y_i &= C_iA\mathbf{x} + C_iB\mathbf{u} = C_iA\mathbf{x} \\
\frac{d^2}{dt^2}y_i &= C_iA^2\mathbf{x} + C_iAB\mathbf{u} = C_iA^2\mathbf{x} \\
&\vdots \\
\frac{d^{d_i}}{dt^{d_i}}y_i &= C_iA^{d_i}\mathbf{x} + C_iA^{d_i-1}B\mathbf{u} = C_iA^{d_i}\mathbf{x} \\
\frac{d^{d_i+1}}{dt^{d_i+1}}y_i &= C_iA^{d_i+1}\mathbf{x} + C_iA^{d_i}B\mathbf{u} \qquad i = 1, 2, \ldots, p
\end{aligned} \tag{3}$$

(This procedure shows us that the decoupling index identifies the first derivative of y_i which contains a $\mathbf{u}$ term.) Adopting the notation that $\hat{\mathbf{y}}$ is the vector whose i-th component is $\frac{d^{d_i+1}}{dt^{d_i+1}}y_i$, G is the matrix whose i-th row is $C_iA^{d_i+1}$ and N is the matrix whose i-th row is $C_iA^{d_i}B$, then Eq (3) can be written in vector-matrix notation as

$$\hat{\mathbf{y}} = G\mathbf{x} + N\mathbf{u} \tag{4}$$

2 Decoupling by Linear Feedback Control

If N^{-1} exists, then by choosing the linear feedback control

$$\mathbf{u} = N^{-1}(-G\mathbf{x} + \mathbf{v}) \tag{5}$$

where $\mathbf{v}$ is a new reference input, Eq (4) becomes

$$\hat{\mathbf{y}} = \mathbf{v} \tag{6}$$

or in terms of each individual input and output

$$\frac{d^{d_i+1}}{dt^{d_i+1}}y_i = v_i \qquad i = 1, 2, \ldots, p \tag{7}$$

(Thus input v_i affects output y_i and it alone). The system of Eq (7) is called the "integrator-decoupled" system as it shows that the input v_i is integrated d_i+1 times to generate y_i. In addition to the invertibility of N being sufficient to achieve this decoupling as shown above, *Falb* & *Wolovich* also showed that it is necessary.

3 Subsequent Control Design

Consider each of the p - SISO "subsystems" described by Eq (7). The order of the i-th subsystem is given by $d_i + 1$. By choosing as state vector for this subsystem

$$\mathbf{x}^i = \begin{pmatrix} y_i \\ \frac{d}{dt} y_i \\ \vdots \\ \frac{d^{d_i}}{dt^{d_i}} y_i \end{pmatrix}$$

we can write its state equation in controllable canonical form (CCF)

$$\frac{d}{dt}\mathbf{x}^i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \mathbf{x}^i + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v_i$$

and use e.g. the standard pole placement algorithm to place its eigenvalues at locations we choose.

4 The Remnant

If we sum the orders of the subsystems, we get

$$q \triangleq \sum_{i=1}^{p} (d_i + 1) = p + \sum_{i=1}^{p} d_i$$

In general $q \leq n$, and if $q < n$ we will have a remnant of order $n - q$. These modes are not present in $\mathbf{y}$, i.e. they are unobservable. Recall for the proper functioning of a system that is not CO, we require it to be detectable, i.e. its unobservable modes should be asymptotically stable. Thus *Falb - Wolovich* decoupling requires detectability of the decoupled system in order to be useful[1].

Substituting Eq (5) into Eq (1) yields

$$\dot{\mathbf{x}} = \left(A - BN^{-1}G\right)\mathbf{x} + BN^{-1}\mathbf{v} \tag{8}$$

[1] There are decoupling approaches which use dynamic compensators in addition to linear feedback which overcome this problem

Since the decoupled system has q eigenvalues at $\lambda = 0$, the characteristic polynomial of $A - BN^{-1}G$ will be $\lambda^q \chi^{rem}(\lambda)$ where $\chi^{rem}(\lambda)$ is the characteristic polynomial of the remnant. Thus the stability of the remnant can be checked by any of the usual means.

5 Discrete-Time LTI Systems

5.1 Decoupling Indices

For the square system

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k) && (9) \\ y_i(k) &= C_i\mathbf{x}(k) \qquad i = 1, 2, \ldots, p && (10) \end{aligned}$$

the decoupling index d_i for output y_i is defined as the **smallest nonnegative integer** such that $C_i A^{d_i} B \neq \mathbf{0}$.
By repeatedly " time advancing" Eq (10), substituting from Eq (9) and using the definition of the decoupling index, we generate the sequence

$$\begin{aligned} y_i(k+1) &= C_iA\mathbf{x}(k) + C_iB\mathbf{u}(k) = C_iA\mathbf{x}(k) \\ y_i(k+2) &= C_iA^2\mathbf{x}(k) + C_iAB\mathbf{u}(k) = C_iA^2\mathbf{x}(k) \\ &\vdots \\ y_i(k+d_i) &= C_iA^{d_i}\mathbf{x}(k) + C_iA^{d_i-1}B\mathbf{u}(k) = C_iA^{d_i}\mathbf{x}(k) \\ y_i(k+d_i+1) &= C_iA^{d_i+1}\mathbf{x}(k) + C_iA^{d_i}B\mathbf{u}(k) \qquad i = 1, 2, \ldots, p && (11) \end{aligned}$$

(This procedure shows us that the decoupling index identifies the first "advance" of y_i which contains a $\mathbf{u}$ term.)
Adopting the notation that $\hat{\mathbf{y}}(k)$ is the vector whose i-th component is $y_i(k + d_i + 1)$, G is the matrix whose i-th row is $C_iA^{d_i+1}$ and N is the matrix whose i-th row is $C_iA^{d_i}B$, then Eq (11) can be written in vector-matrix notation as

$$\hat{\mathbf{y}}(k) = G\mathbf{x}(k) + N\mathbf{u}(k) \qquad (12)$$

5.2 Decoupling by Linear feedback Control

If N^{-1} exists, then by choosing the linear feedback control

$$\mathbf{u}(k) = N^{-1}\left(-G\mathbf{x}(k) + \mathbf{v}(k)\right) \qquad (13)$$

where $\mathbf{v}(k)$ is a new reference input, Eq (12) becomes

$$\hat{\mathbf{y}}(k) = \mathbf{v}(k) \tag{14}$$

or in terms of each individual input and output

$$y_i(k + d_1 + 1) = v_i(k) \qquad\qquad i = 1, 2, \ldots, p \tag{15}$$

(Thus input v_i affects output y_i and it alone). The system of Eq (14) is called the "delay-decoupled" system as it shows that the input v_i is time delayed $d_i + 1$ times to generate y_i. In addition to the invertibility of N being sufficient to achieve this decoupling as shown above, it has been shown that it is also necessary.

5.3 Subsequent Control Design

Unlike the continuous-time case, the automatic placing of all decoupled poles at 0 may not be viewed as necessarily "bad" as it generates dead-beat controllers. If some redesign is deemed necessary, we may proceed as for the continuous-time case.

5.4 The Remnant

As with the continuous-time case, delay-decoupling usually also involves a remnant of order $q = \sum_{i=1}^{p}(d_1 + 1) < n$. For the correct working of the *Falb - Wolovich*-style decoupling, the system must be detectable, i.e. the remnant must be asymptotically stable. Substituting Eq (13) into Eq (9) yields

$$\mathbf{x}(k + 1) = \left(A - BN^{-1}G\right)\mathbf{x}(k) + BN^{-1}\mathbf{v}(k) \tag{16}$$

Since the decoupled system has q eigenvalues at $\lambda = 0$, the characteristic polynomial of $A - BN^{-1}G$ will be $\lambda^q \chi^{rem}(\lambda)$ where $\chi^{rem}(\lambda)$ is the characteristic polynomial of the remnant. Thus the stability of the remnant can be checked by any of the usual means.