1. Applying the algorithm:

Step 1.

$$ad_{\mathbf{f}}\,\mathbf{g} \triangleq [\mathbf{f},\mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g} = \mathbf{0} - \begin{pmatrix} 0 & 1 & 0 \\ \cos x_1 & 0 & 1 \\ 0 & -2 & -5 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}$$

$$ad_{\mathbf{f}^2}\mathbf{g} = [\mathbf{f}, ad_{\mathbf{f}}\mathbf{g}] = \frac{\partial ad_{\mathbf{f}}\,\mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}} ad_{\mathbf{f}}\mathbf{g} = \mathbf{0} - \begin{pmatrix} 0 & 1 & 0 \\ \cos x_1 & 0 & 1 \\ 0 & -2 & -5 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 23 \end{pmatrix}$$

Hence

$$F_3 = \{\mathbf{g},\ ad_{\mathbf{f}}\,\mathbf{g},\ ad_{\mathbf{f}^2}\,\mathbf{g}\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ 23 \end{pmatrix} \right\}.$$

Step 2. F_3 is by inspection linearly independent, and $F_2 = \{\mathbf{g}, ad_{\mathbf{f}}\,\mathbf{g}\}$ is involutive (why ?). So ....

Step 3.

$$L_{\mathbf{g}} z_1 = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad \frac{\partial z_1}{\partial x_3} = 0 \tag{1}$$

$$L_{ad_{\mathbf{f}}\,\mathbf{g}}\ z_1 = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix} = 0$$

$$\Rightarrow \quad -\frac{\partial z_1}{\partial x_2} + 5\frac{\partial z_1}{\partial x_3} = 0 \tag{2}$$

$$L_{ad_{\mathbf{f}^2}\,\mathbf{g}}\ z_1 \neq 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} 1 \\ -5 \\ 23 \end{pmatrix} \neq 0$$

$$\Rightarrow \quad \frac{\partial z_1}{\partial x_1} - 5\frac{\partial z_1}{\partial x_2} + 23\frac{\partial z_1}{\partial x_3} \neq 0$$

$$\Rightarrow \quad \frac{\partial z_1}{\partial x_1} - 5\frac{\partial z_1}{\partial x_2} + 23\frac{\partial z_1}{\partial x_3} = 1 \ \text{(say)} \tag{3}$$

A solution of Eqs (1), (2) and (3) is

$$z_1 = x_1$$

Step 4.

$$
\begin{aligned}
z_2 &= L_{\mathbf{f}}\ z_1 \\
&= \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} x_2 \\ \sin x_1 + x_3 \\ -2x_2 - 5x_3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ \sin x_1 + x_3 \\ -2x_2 - 5x_3 \end{pmatrix} \\
&= x_2
\end{aligned}
$$

$$
\begin{aligned}
z_3 &= L_{\mathbf{f}^2}\ z_1 = L_{\mathbf{f}}\ z_2 \\
&= \begin{pmatrix} \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_3} \end{pmatrix} \begin{pmatrix} x_2 \\ \sin x_1 + x_3 \\ -2x_2 - 5x_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ \sin x_1 + x_3 \\ -2x_2 - 5x_3 \end{pmatrix} \\
&= \sin x_1 + x_3
\end{aligned}
$$

We compute

$$
\begin{aligned}
L_{\mathbf{g}}\ z_3 &= \begin{pmatrix} \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos x_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= 1
\end{aligned}
$$

and

$$
\begin{aligned}
L_{\mathbf{f}}\ z_3 &= \begin{pmatrix} \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} x_2 \\ \sin x_1 + x_3 \\ -2x_2 - 5x_3 \end{pmatrix} \\
&= \begin{pmatrix} \cos x_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ \sin x_1 + x_3 \\ -2x_2 - 5x_3 \end{pmatrix} \\
&= x_2 \cos x_1 - 2x_2 - 5x_3
\end{aligned}
$$

then

$$
\begin{aligned}
\beta(\mathbf{x}) &= \frac{1}{L_{\mathbf{g}}\ z_3} \\
&= 1
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha(\mathbf{x}) &= -\frac{L_{\mathbf{f}}\ z_3}{L_{\mathbf{g}}\ z_3} \\
&= -\left(x_2 \cos x_1 - 2x_2 - 5x_3\right)
\end{aligned}
$$

and the feedback controller that input -state linearises the system is
$u = -(x_2 \cos x_1 - 2x_2 - 5x_3) + v$.
The control $v = k_1 z_1 + k_2 z_2 + k_3 z_3 + w$ applied to the CCF linear system leads to the characteristic polynomial $\lambda^3 - k_3\lambda^2 - k_2\lambda - k_1$. For asymptotic stability, we need to choose $k_3 < 0$, $k_1 < 0$ and $k_1 + k_2 k_3 > 0$.[1] If we decide to place all three poles at -1, say, then $k_1 = -1, k_2 = -3$ and $k_3 = -3$. The control applied to the original system is then

$$\begin{aligned} u &= -(x_2 \cos x_1 - 2x_2 - 5x_3) + v \\ &= -(x_2 \cos x_1 - 2x_2 - 5x_3) - z_1 - 3z_2 - 3z_3 + w \\ &= -(x_2 \cos x_1 - 2x_2 - 5x_3) - x_1 - 3x_2 - 3(\sin x_1 + x_3) + w \\ &= -x_1 - x_2 + 2x_5 - 3\sin x_1 - x_2 \cos x_1 + w. \end{aligned}$$

[1]These conditions come from *Routh* criterion.

2. Applying the algorithm:

Step 1.

$$\begin{aligned}
ad_{\mathbf{f}}\,\mathbf{g} = [\mathbf{f},\mathbf{g}] &= \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g} \\
&= \begin{pmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \sin x_1 \\ \sin x_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \cos x_1 & 0 & 0 \\ 0 & \cos x_2 & 0 \end{pmatrix}\begin{pmatrix} \cos x_3 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sin x_2 \sin x_3 \\ -\cos x_1 \cos_3 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
ad_{\mathbf{f}^2}\,\mathbf{g} = [\mathbf{f}, ad_{\mathbf{f}}\,\mathbf{g}] &= \frac{\partial ad_{\mathbf{f}}\,\mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}} ad_{\mathbf{f}}\,\mathbf{g} \\
&= \begin{pmatrix} 0 & -\cos x_2 \sin x_3 & -\sin x_2 \cos x_3 \\ \sin x_1 \cos x_3 & 0 & \cos x_1 \sin x_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \sin x_1 \\ \sin x_2 \end{pmatrix} \\
&\quad - \begin{pmatrix} 0 & 0 & 0 \\ \cos x_1 & 0 & 0 \\ 0 & \cos x_2 & 0 \end{pmatrix}\begin{pmatrix} -\sin x_2 \sin x_3 \\ -\cos x_1 \cos x_3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sin x_1 \cos x_2 \sin x_3 - \sin^2 x_2 \cos x_3 \\ 2\cos x_1 \sin x_2 \sin x_3 \\ \cos x_1 \cos x_2 \cos x_3 \end{pmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
F_3 &= \{\mathbf{g},\ ad_{\mathbf{f}}\,\mathbf{g},\ ad_{\mathbf{f}^2}\,\mathbf{g}\} \\
&= \left\{ \begin{pmatrix} \cos x_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin x_2 \sin x_3 \\ -\cos x_1 \cos x_3 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin x_1 \cos x_2 \sin x_3 - \sin^2 x_2 \cos x_3 \\ 2\cos x_1 \sin x_2 \sin x_3 \\ \cos x_1 \cos x_2 \cos x_3 \end{pmatrix} \right\}.
\end{aligned}$$

Step 2. F_3 is linearly independent provided $\cos x_1 \cos x_2 \cos x_3 \neq 0$. In particular there is a neighbourhood of $\mathbf{x}_e = \mathbf{0}$ where this is true. $F_2 = \{\mathbf{g},\ ad_{\mathbf{f}}\,\mathbf{g}\}$ is involutive provided $\cos x_1 \neq 0$ since

$$\begin{aligned}
[ad_{\mathbf{f}}\,\mathbf{g}, \mathbf{g}] &= \frac{\partial \mathbf{g}}{\partial \mathbf{x}} ad_{\mathbf{f}}\,\mathbf{g} - \frac{\partial ad_{\mathbf{f}}\,\mathbf{g}}{\mathbf{x}}\mathbf{g} \\
&= \begin{pmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\sin x_2 \sin x_3 \\ -\cos x_1 \cos x_3 \\ 0 \end{pmatrix} \\
&\quad - \begin{pmatrix} 0 & -\cos x_2 \sin x_3 & -\sin x_2 \cos x_3 \\ \sin x_1 \cos x_3 & 0 & \cos x_1 \sin x_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos x_3 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -\sin x_1 \cos^2 x_3 \\ 0 \end{pmatrix} \\
&= \left(\frac{\sin x_1 \sin x_2 \sin x_3}{\cos x_1}\right)\mathbf{g} + \left(\frac{\sin x_1 \cos x_3}{\cos x_1}\right) ad_{\mathbf{f}}\,\mathbf{g}
\end{aligned}$$

So ....

Step 3.

$$L_{\mathbf{g}} z_1 = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} \cos x_3 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad \cos x_3 \frac{\partial z_1}{\partial x_1} = 0 \tag{4}$$

$$L_{ad_{\mathbf{f}}\,\mathbf{g}}\ z_1 = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} -\sin x_2 \sin x_3 \\ -\cos x_1 \cos x_3 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow \quad -\sin x_1 \sin x_3 \frac{\partial z_1}{\partial x_1} - \cos x_1 \cos x_3 \frac{\partial z_1}{\partial x_2} = 0 \tag{5}$$

$$L_{ad_{\mathbf{f}^2}\,\mathbf{g}}\ z_1 \neq 0$$

$$\Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} -\sin x_1 \cos x_2 \sin x_3 - \sin^2 x_2 \cos x_3 \\ 2\cos x_1 \sin x_2 \sin x_3 \\ \cos x_1 \cos x_2 \cos x_3 \end{pmatrix} \neq 0$$

$$\begin{aligned} \Rightarrow \quad & -(\sin x_1 \cos x_2 \sin x_3 + \sin^2 x_2 \cos x_3) \frac{\partial z_1}{\partial x_1} \\ & +2\cos x_1 \sin x_2 \sin x_3 \frac{\partial z_1}{\partial x_2} + \cos x_1 \cos x_2 \cos x_3 \frac{\partial z_1}{\partial x_3} \neq 0 \\ \Rightarrow \quad & -(\sin x_1 \cos x_2 \sin x_3 + \sin^2 x_2 \cos x_3) \frac{\partial z_1}{\partial x_1} \\ & +2\cos x_1 \sin x_2 \sin x_3 \frac{\partial z_1}{\partial x_2} + \cos x_1 \cos x_2 \cos x_3 \frac{\partial z_1}{\partial x_3} = \cos x_1 \cos x_2 \cos x_3 \\ & \text{(say)} \end{aligned} \tag{6}$$

A solution of Eqs (4), (5) and (6) for $\cos x_1 \cos x_2 \cos x_3 \neq 0$ is

$$z_1 = x_3$$

Step 4.

$$\begin{aligned} z_2 &= L_{\mathbf{f}}\ z_1 \\ &= \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \sin x_1 \\ \sin x_2 \end{pmatrix} \\ &= \sin x_2 \end{aligned}$$

$$
\begin{aligned}
z_3 &= L_{\mathbf{f}^2}\ z_1 = L_{\mathbf{f}}\ z_2 \\
&= \begin{pmatrix} \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_3} \end{pmatrix} \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \cos x_2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sin x_1 \\ \sin x_2 \end{pmatrix} \\
&= \sin x_1 \cos x_2
\end{aligned}
$$

We compute

$$
\begin{aligned}
L_{\mathbf{g}}\ z_3 &= \begin{pmatrix} \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ g_3(\mathbf{x}) \end{pmatrix} \\
&= \begin{pmatrix} \cos x_1 \cos x_2 & -\sin x_1 \sin x_2 & 0 \end{pmatrix} \begin{pmatrix} \cos x_3 \\ 0 \\ 0 \end{pmatrix} \\
&= \cos x_1 \cos x_2 \cos x_3
\end{aligned}
$$

and

$$
\begin{aligned}
L_{\mathbf{f}}\ z_3 &= \begin{pmatrix} \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{pmatrix} \\
&= \begin{pmatrix} \cos x_1 \cos x_2 & -\sin x_1 \sin x_2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sin x_1 \\ \sin x_2 \end{pmatrix} \\
&= -\sin^2 x_1 \sin x_2
\end{aligned}
$$

then

$$
\begin{aligned}
\beta(\mathbf{x}) &= \frac{1}{L_{\mathbf{g}}\ z_3} \\
&= \frac{1}{\cos x_1 \cos x_2 \cos x_3}
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha(\mathbf{x}) &= -\frac{L_{\mathbf{f}}\ z_3}{L_{\mathbf{g}}\ z_3} \\
&= \frac{\sin^2 x_1 \sin x_2}{\cos x_1 \cos x_2 \cos x_3}
\end{aligned}
$$

and the feedback controller that input -state linearises the system is

$$
u = \frac{\sin^2 x_1 \sin x_2}{\cos x_1 \cos x_2 \cos x_3} + \frac{v}{\cos x_1 \cos x_2 \cos x_3}
$$

. The transformation

$$\begin{aligned} \mathbf{z} &= \Phi(\mathbf{x}) = \begin{pmatrix} x_3 \\ \sin x_2 \\ \sin x_1 \cos x_2 \end{pmatrix} \\ \Leftrightarrow \mathbf{x} &= \Phi^{-1}(\mathbf{z}) = \begin{pmatrix} \arcsin\left(\frac{z_3}{\sqrt{1-z_2^2}}\right) \\ \arcsin z_2 \\ z_1 \end{pmatrix} \end{aligned}$$

is valid in a neighbourhood of $\mathbf{0}$ where the appropriate branch of $\arcsin x$ is well defined, e.g. $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

3. Applying the algorithm:

Step 1.

$$ad_{\mathbf{f}}\,\mathbf{g} = [\mathbf{f}, \mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g} = \mathbf{0} - \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$ad_{\mathbf{f}^2}\,\mathbf{g} = [\mathbf{f}, ad_{\mathbf{f}}\,\mathbf{g}] = \frac{\partial ad_{\mathbf{f}}\,\mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}} ad_{\mathbf{f}}\,\mathbf{g} = \mathbf{0} - \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$$

Hence

$$F_3 = \{\mathbf{g},\ ad_{\mathbf{f}}\,\mathbf{g},\ ad_{\mathbf{f}^2}\,\mathbf{g}\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \right\}.$$

Step 2. F_3 is linearly independent, and $F_2 = \{\mathbf{g},\ ad_{\mathbf{f}}\,\mathbf{g}\}$ is involutive (why ?). So ....

Step 3.

$$\begin{aligned} L_{\mathbf{g}} z_1 = 0 \quad &\Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \\ &\Rightarrow \quad \frac{\partial z_1}{\partial x_1} = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} L_{ad_{\mathbf{f}}\,\mathbf{g}}\, z_1 = 0 \quad &\Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = 0 \\ &\Rightarrow \quad -\frac{\partial z_1}{\partial x_1} - \frac{\partial z_1}{\partial x_3} = 0 \end{aligned} \tag{8}$$

$$\begin{aligned} L_{ad_{\mathbf{f}^2}\,\mathbf{g}}\, z_1 \neq 0 \quad &\Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \neq 0 \\ &\Rightarrow \quad \frac{\partial z_1}{\partial x_2} + 4\frac{\partial z_1}{\partial x_3} \neq 0 \\ &\Rightarrow \quad \frac{\partial z_1}{\partial x_2} + 4\frac{\partial z_1}{\partial x_3} = 1 \ \text{(say)} \end{aligned} \tag{9}$$

A solution of Eqs (7), (8) and (9) is

$$z_1 = x_2$$

Step 4.

$$\begin{aligned} z_2 &= L_{\mathbf{f}}\ z_1 \\ &= \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3} \end{pmatrix} \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 - x_3 \\ \sin x_3 \\ x_1 - 2x_2 + 3x_3 \end{pmatrix} \\ &= x_3 \end{aligned}$$

$$\begin{aligned} z_3 &= L_{\mathbf{f}^2}\ z_1 = L_{\mathbf{f}}\ z_2 \\ &= \begin{pmatrix} \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} & \frac{\partial z_2}{\partial x_3} \end{pmatrix} \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 - x_3 \\ x_3 \\ x_1 - 2x_2 + 3x_3 \end{pmatrix} \\ &= x_1 - 2x_2 + 3x_3 \end{aligned}$$

We compute

$$\begin{aligned} L_{\mathbf{g}}\ z_3 &= \begin{pmatrix} \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ g_3(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} L_{\mathbf{f}}\ z_3 &= \begin{pmatrix} \frac{\partial z_3}{\partial x_1} & \frac{\partial z_3}{\partial x_2} & \frac{\partial z_3}{\partial x_3} \end{pmatrix} \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 - x_3 \\ x_3 \\ x_1 - 2x_2 + 3x_3 \end{pmatrix} \\ &= 4x_1 - 6x_2 + 6x_3 \end{aligned}$$

then

$$\begin{aligned} \beta(\mathbf{x}) &= \frac{1}{L_{\mathbf{g}}\ z_3} \\ &= 1 \end{aligned}$$

and

$$
\begin{aligned}
\alpha(\mathbf{x}) &= -\frac{L_{\mathbf{f}}\ z_3}{L_{\mathbf{g}}\ z_3} \\
&= -(4x_1 - 6x_2 + 6x_3)
\end{aligned}
$$

and the feedback controller that input -state linearises the system is $u = -(4x_1 - 6x_2 + 6x_3) + v$.
Placing the C-L poles at -1, -2 and -3 gives a characteristic polynomial of $\lambda^3+6\lambda^2+11\lambda+6$ which means that v should be chosen as $v = -6z_1 - 11z_2 - 6z_3 + w$.
This leads to the control to be applied to the original system being

$$
\begin{aligned}
u &= -(4x_1 - 6x_2 + 6x_3) + v \\
&= -(4x_1 - 6x_2 + 6x_3) - 6z_1 - 11z_2 - 6z_3 + w \\
&= u = -(4x_1 - 6x_2 + 6x_3) - 6x_2 - 11x_3 - 6(x_1 - 2x_2 + 3x_3) + w \\
&= -10x_1 + 12x_2 - 35x_3 + w.
\end{aligned}
$$

There is no (essential) difference between this approach and the pole placement algorithm for LTI systems discussed earlier. For the affine LTI system $\dot{\mathbf{x}} = A\mathbf{x} + bu$

- $$F_n = \{\mathbf{g},\ ad_{\mathbf{f}}\ \mathbf{g}, ad_{\mathbf{f}^2}\ \mathbf{g}, \ldots, ad_{\mathbf{f}^{n-1}}\ \mathbf{g}\} = \{b, -Ab, A^2b, \ldots, (-1)^n A^{n-1}b\}$$

  Thus linear independence of F_n corresponds to the controllability matrix U being of full rank. Any set of constant vector fields in involutive, and hence so is F_{n-1}.

- Here we compute the transformation $\mathbf{z} = \Phi(\mathbf{x})$. Because of the LTI nature of the vector fields $\Phi(\mathbf{x}) = P\mathbf{x}$. If we apply the transformation $\mathbf{z} = P\mathbf{x}$ without applying the "cancelling "control $u = \alpha(\mathbf{x}) + \beta\mathbf{x})v$. we get the CCF system

  $$
  \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ \mathbf{ax} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u
  $$

  where $\mathbf{a} = [-a_0, -a_1, \ldots, -a_{n-1}]$ consists of the coefficients of the characteristic polynomial of A. When we discussed LTI pole placement, we initially computed T ($\mathbf{x} = T\mathbf{z}$), where T transformed the system to CCF. Hence $P = T^{-1}$. (recall that we needed T^{-1} to return to the original state representation, having done the design with the CCF system.) If we view the linear control law obtained by the pole placement algorithm as comprised of a part designed to cancel the O-L characteristic polynomial followed by a part designed to insert the desired C-L characteristic polynomial we see the direct connection between the two approaches.