1.
$$\dot{\mathbf{x}} = \begin{pmatrix} x_1 + x_2^2 \\ x_1 + x_2^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$
$$y = x_2$$

(a) **Computing the relative degree**:

$$\begin{aligned} y &= x_2 \\ \Rightarrow \dot{y} &= \dot{x}_2 = x_1 + x_2^3 \\ \Rightarrow \ddot{y} &= \dot{x}_1 + 3x_2^2\dot{x}_2 = x_1 + x_2^2 + u + 3x_2^2(x_1 + x_2^3) \\ &= \underbrace{x_1 + x_2^2 + 3x_1x_2^2 + 3x_2^5}_{L_{\mathbf{f}^2}h(x_1,x_2)=A(x_1,x_2)} \qquad + \underbrace{1}_{L_{\mathbf{g}}L_{\mathbf{f}}h(x_1,x_2)=B(x_1,x_2)} \qquad u \end{aligned}$$

Therefore $r = 2$. Hence $r = n$ and so there is no remnant.
Describing the Normal Form:

$$\begin{aligned} \xi_1 = \mathbf{\Phi}_1(\mathbf{x}) = \quad h(\mathbf{x}) &= x_2 \\ \xi_2 = \mathbf{\Phi}_2(\mathbf{x}) = L_{\mathbf{f}}h(\mathbf{x}) &= x_1 + x_2^3 \end{aligned}$$

Solving to get the inverse transformation

$$\begin{aligned} \Rightarrow x_1 = \mathbf{\Phi}_1^{-1}(\xi) &= \xi_2 - \xi_1^3 \\ x_2 = \mathbf{\Phi}_2^{-1}(\xi) &= \xi_1 \end{aligned}$$

The normal form is

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= A + Bu \\ y &= \xi_1 \end{aligned}$$

(b) **I/O Linearisation Feedback**
With
$$u = \frac{-A + v}{B} = -(x_1 + x_2^2 + 3x_1x_2^2 + 3x_2^5) + v$$
the linearised system is

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v \\ y &= \xi_1 \end{aligned}$$

(c) To place the poles at $-1, -2$, the desired closed loop characteristic polynomial is $\chi_{cl}(\lambda) = \lambda^2 + 3\lambda + 2$ which leads to the feedback control

$$v = -2\xi_1 - 3\xi_2 + w.$$

This in turn gives

$$\begin{aligned} u &= -(x_1 + x_2^2 + 3x_1x_2^2 + 3x_2^5) + v \\ &= -x_1 - x_2^2 - 3x_1x_2^2 - 3x_2^5 - 2\xi_1 - 3\xi_2 + w \\ &= -x_1 - x_2^2 - 3x_1x_2^2 - 3x_2^5 - 2x_2 - 3(x_1 + x_2^3) + w \\ &= -4x_1 - 2x_2 - x_2^2 - 3x_1x_2^2 - 3x_2^3 - 3x_2^5 + w \end{aligned}$$

(d) ***Jacobian* linearised system**: The system linearised about $\mathbf{x}_e = \mathbf{0}, u_e = 0$ is

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

This system is completely controllable, and application of the standard pole placement algorithm to place the poles at -1,-2 yields the control law

$$u = -4x_1 - 2x_2 + w$$

(verify this by finding the characteristic equation of the closed loop system.)

2.

$$\dot{\mathbf{x}} = \begin{pmatrix} x_1 + x_2^2 \\ x_1 + x_2^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$
$$y = x_2$$

(a) **Computing the relative degree**

$$\begin{aligned} y &= x_1 \\ \Rightarrow \dot{y} &= \dot{x}_1 = \underbrace{x_1 + x_2^2}_{L_{\mathbf{f}}h(x_1,x_2)=A(x_1,x_2)} + \underbrace{1}_{L_{\mathbf{g}}h(x_1,x_2)=B(x_1,x_2)} u \end{aligned}$$

Therefore $r = 1$. Hence $r < n$ and so there is a remnant.
Describing the Normal Form

$$\begin{aligned} \xi_1 = \mathbf{\Phi}_1(\mathbf{x}) = h(\mathbf{x}) &= x_1 \\ \eta_2 = \mathbf{\Phi}_2(\mathbf{x}) & \\ \Rightarrow \dot{\eta}_2 = \frac{\partial \mathbf{\Phi}_2}{\partial \mathbf{x}}(\dot{\mathbf{x}}) &= \frac{\partial \mathbf{\Phi}_2}{\partial \mathbf{x}}(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u) \\ &= \underbrace{\frac{\partial \mathbf{\Phi}_2}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})}_{L_{\mathbf{f}}\mathbf{\Phi}_2(\mathbf{x})} + \underbrace{\frac{\partial \mathbf{\Phi}_2}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})}_{L_{\mathbf{g}}\mathbf{\Phi}_2(\mathbf{x})} u \end{aligned}$$

It is known that $\mathbf{\Phi}_2$ can be chosen so that $L_{\mathbf{g}}\mathbf{\Phi}_2(\mathbf{x}) = 0$, and in this problem we calculate that $L_{\mathbf{g}}\mathbf{\Phi}_2(\mathbf{x}) = \frac{\partial \mathbf{\Phi}_2}{\partial x_1}$ and so $\mathbf{\Phi}_2(\mathbf{x}) = C(x_2)$, where C is a function of its argument. Since $\eta_2 = \mathbf{\Phi}_2(\mathbf{x}) = C(x_2)$, C must also be invertible. Solving to get the inverse transformation

$$\begin{aligned} \Rightarrow x_1 = \mathbf{\Phi}_1^{-1}(\xi) &= \xi_1 \\ x_2 = \mathbf{\Phi}_2^{-1}(\xi) &= C^{-1}(\eta_2) \end{aligned}$$

The normal form is

$$\begin{aligned} \dot{\xi}_1 &= A + Bu = x_1 + x_2^2 + u \\ \dot{\eta}_2 &= \underbrace{\frac{dC}{dx_2}(x_1 + x_2^3)}_{q_2(\mathbf{x})} \\ y &= \xi_1 \end{aligned}$$

(b) **I/O Linearisation Feedback**
With

$$u = \frac{-A + v}{B} = -x_1 - x_2^2 + v$$

the linearised system is

$$\begin{aligned} \dot{\xi}_1 &= v \\ \dot{\eta}_2 &= q_2(\mathbf{x}) \\ y &= \xi_1 \end{aligned}$$

(c) **Zero Dynamics**
For this example

$$y \equiv 0 \Leftrightarrow x_1 \equiv 0.$$

Hence the zero dynamics are (see Normal form above)

$$\dot{\eta}_2 = \frac{dC}{dx_2}[x_2^3] = C'(C^{-1}(\eta_2))[C^{-1}(\eta_2)]^3$$

Prove that this is unstable for all invertible C, e.g. $C(x) = x$ gives $\dot{\eta}_2 = \eta_2^3$. What does this imply about the use of I/O feedback linearisation for this control system?

3.

$$\dot{\mathbf{x}} = \begin{pmatrix} x_1 + x_2^2 \\ x_1 + x_2^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

From Q.1, we know that choosing the output equation as $\tilde{h}(\mathbf{x}) = x_2$ will solve the problem. But we'll pretend that we don't know this and proceed as follows: Apply the algorithm:

Step 1.

$$[\mathbf{f}, \mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g} = \mathbf{0} - \begin{pmatrix} 1 & 2x_2 \\ 1 & 3x_2^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Hence

$$F_2 = \{\mathbf{g},\ ad_{\mathbf{f}}\mathbf{g}\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}.$$

Step 2. F_2 is by inspection linearly independent, and $F_1 = \{\mathbf{g}\}$ is involutive. So ....

Step 3.

$$\begin{aligned} L_{\mathbf{g}} z_1 = 0 \quad &\Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \\ &\Rightarrow \quad \frac{\partial z_1}{\partial x_1} = 0 \qquad (1) \end{aligned}$$

$$\begin{aligned} L_{ad_{\mathbf{f}}\mathbf{g}}\, z_1 \neq 0 \quad &\Rightarrow \quad \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \neq 0 \\ &\Rightarrow \quad -\frac{\partial z_1}{\partial x_1} - \frac{\partial z_1}{\partial x_2} \neq 0 \qquad (2) \end{aligned}$$

To satisfy this last inequality (2), choose

$$\frac{\partial z_1}{\partial x_1} + \frac{\partial z_1}{\partial x_2} = 1, \tag{3}$$

for instance. A solution of Eqs (1) and (3) is

$$z_1 = x_2$$

and this will lead to the same solution as Q.1
More generally, the system (1) - (2) has solution $z_1 = F(x_2)$ where $F'(x_2) \neq 0$. We'll continue the algorithm using this.

Step 4.

$$\begin{aligned} z_2 = L_{\mathbf{f}} z_1 &= \frac{\partial z_1}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) \\ &= \begin{bmatrix} 0 & F'(x_2) \end{bmatrix} \begin{bmatrix} x_1 + x_2^2 \\ x_1 + x_2^3 \end{bmatrix} = F'(x_2)[x_1 + x_2^3] \end{aligned} \tag{4}$$

To compute the linearising control we need [1]

$$\begin{aligned} B = L_{\mathbf{g}} z_2 &= \frac{\partial z_2}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}) \\ &= \begin{bmatrix} F'(x_2), & F''(x_2)[x_1 + x_2^3] + F'(x_2)[3x_2^2] \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= F'(x_2) \end{aligned}$$

$$\begin{aligned} A = L_{\mathbf{f}} z_2 &= \frac{\partial z_2}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) \\ &= \begin{bmatrix} F'(x_2), & F''(x_2)[x_1 + x_2^3] + F'(x_2)[3x_2^2] \end{bmatrix} \begin{bmatrix} x_1 + x_2^2 \\ x_1 + x_2^3 \end{bmatrix} \\ &= F'(x_2)[x_1 + x_2^2 + 3x_1 x_2^2 + 3x_2^5] + F''(x_2)[x_1 + x_2^3]^2 \end{aligned}$$

and the control itself is defined by

$$A + Bu = v \qquad \Leftrightarrow \qquad u = \frac{-A + v}{B}$$

If we take the concrete example $F(x_2) = e^{x_2} - 1$ (why the -1 ?), we get the state transformation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{x_2} - 1 \\ e^{x_2}(x_1 + x_2^3) \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{z_2}{1+z_1} - (\ln(1+z_1))^3 \\ \ln(1+z_1) \end{pmatrix}$$

and the control

$$u = -x_1 - x_1^2 - x_2^2 - 3x_1 x_2^2 - 2x_1 x_2^3 - 3x_2^5 - x_2^6 + e^{-x_2} v$$

transforms the system to *Brunovsky* canonical form.

[1] we can also do this by computing $\dot{z}_2$