1. (a) The *Lyapunov* equation is $A^TP + PA = -Q$. We'll use $Q = I$ in each of the following:

(a-1)

$$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$$\begin{aligned} -2p_2 &= -1 \\ p_1 - 2p_2 - p_3 &= 0 \\ 2p_2 - 4p_3 &= -1 \end{aligned}$$

which gives

$$P = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

which is positive definite (leading principal minors are 3/2 and 1/2), and so $\mathbf{x}_e = \mathbf{0}$ is asymptotically stable.

(a-2)

$$\begin{pmatrix} -1 & -0.5 \\ -2 & 0.5 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$$\begin{aligned} -2p_1 - p_2 &= -1 \\ 2p_1 - 0.5p_2 - 0.5p_3 &= 0 \\ 4p_2 + p_3 &= -1 \end{aligned}$$

which gives

$$P = \begin{pmatrix} 2 & -3 \\ -3 & 11 \end{pmatrix}$$

which is positive definite (leading principal minors are 2 and 13), and so $\mathbf{x}_e = \mathbf{0}$ is asymptotically stable.

(a-3)

$$\begin{pmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{pmatrix}\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

or

$$\begin{aligned} -12p_3 &= -1 \\ p_1 - 11p_3 - 6p_5 &= 0 \\ p_2 - 6p_3 - 6p_6 &= 0 \\ 2p_2 - 22p_5 &= -1 \\ p_3 + p_4 - 6p_5 - 11p_6 &= 0 \\ 2p_5 - 12p_6 &= -1 \end{aligned}$$

which gives[1]

$$P = \frac{1}{120}\begin{pmatrix} 218 & 138 & 10 \\ 138 & 191 & 18 \\ 10 & 18 & 13 \end{pmatrix}$$

which is positive definite (leading principal minors are 218/120, 22594/14400 and 76730/1728000), and so $\mathbf{x}_e = \mathbf{0}$ is asymptotically stable.

(b) The Discrete *Lyapunov* equation is $A^T PA - P = -Q$. We'll use $Q = I$ in each of the following:

(b-1)

$$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$$\begin{aligned} p_3 - p_1 &= -1 \\ -2p_2 + 2p_3 &= 0 \\ p_1 - 4p_2 + 3p_3 &= -1 \end{aligned}$$

which gives no (consistent) solution for P. Thus P is not p.d. and so $\mathbf{x}_e = \mathbf{0}$ is not asymptotically stable.

(b-2)

$$\begin{pmatrix} -1 & -0.5 \\ 2 & 0.5 \end{pmatrix}\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -0.5 & 0.5 \end{pmatrix} - \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$$\begin{aligned} p_2 + 0.25p_3 &= -1 \\ -2p_1 - 2.5p_2 - 0.25p_3 &= 0 \\ 4p_1 + 2p_2 - 0.75p_3 &= -1 \end{aligned}$$

which gives

$$P = \begin{pmatrix} 2.75 & -3 \\ -3 & 8 \end{pmatrix}$$

which is positive definite (leading principal minors are 2.75 and 13), and so $\mathbf{x}_e = \mathbf{0}$ is asymptotically stable.

2. Using the approach of "Two Canonical Examples: 2-d case", we have

(a)

$$\ddot{x} - \dot{x} + x^2 = u$$

Suppose that $u = u_1(x) + u_2(\dot{x})$ which gives

$$\ddot{x} + \underbrace{(-\dot{x} - u_2(\dot{x}))}_{b(\dot{x})} + \underbrace{x^2 - u_1(x)}_{c(x)} = 0$$

We require that b and c be continuous and that they have the same sign as their arguments, i.e.

$$\begin{aligned} \dot{x}b(\dot{x}) &> 0, & \dot{x} \neq 0 \\ xc(x) &> 0, & x \neq 0 \end{aligned}$$

[1]Note that having got p_3 from the first equation, the third, fourth and sixth equations can be solved for p_2, p_5, and p_6. Then p_1 and p_4 can be read off from the second and fifth equations respectively.

This can be achieved by choosing e.g. $u_2(\dot{x}) = -2\dot{x}$ and $u_1(x) = x^2 - x$. This choice for u_1 involves a nonlinearity cancellation; Is this wise? Perhaps a better strategy is to choose a structure for u_1 which *dominates* the nonlinearity rather than cancels it; this leads to a more robust controller. For example if $u_1 = -K(x + x^3)$ (and u_2 as before) then the system equation becomes

$$\begin{aligned} \ddot{x} + \dot{x} + Kx + x^2 + Kx^3 &= 0 \\ \Rightarrow \ddot{x} + \underbrace{\dot{x}}_{b(\dot{x})} + \underbrace{Kx(1 + \frac{1}{K}x + x^2)}_{c(x)} &= 0 \end{aligned}$$

Then b and c have the required properties provided K is chosen so that $0 < \frac{1}{2K} < 1$.

(b)

$$\ddot{x} + \beta\dot{x}^3 + \alpha x = u$$

where it is known that $|\beta| < 2$ and $4 < \alpha < 6$.
We can choose $u = -2\dot{x}^3$ giving

$$\ddot{x} + \underbrace{(\beta + 2)\dot{x}^3}_{b(\dot{x})} + \underbrace{\alpha x}_{c(x)} = 0$$

. Why isn't it necessary to modify the x-term ? What would happen if $-6 < \alpha < -4$?

(c)

$$\ddot{x} + \dot{x}^5 = x^2 u$$

If we choose $u = -x$, the system becomes

$$\ddot{x} + \underbrace{\dot{x}^5}_{b(\dot{x})} + \underbrace{x^3}_{c(x)} = 0$$

3. (a) V is p. d. since

$$V = \mathbf{x}^T P \mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and P is a positive definite matrix.
Secondly

$$\begin{aligned} V(\mathbf{x}) &= \frac{1}{2}x_1^2 + x_1x_2 + \frac{3}{2}x_2^2 \\ \Rightarrow \frac{dV}{dt} &= x_1\dot{x}_1 + \dot{x}_1x_2 + x_1\dot{x}_2 + 3x_2\dot{x}_2 \\ &= (x_1 + x_2)\dot{x}_1 + (x_1 + 3x_2)\dot{x}_2 \\ &= (x_1 + x_2)x_2 + (x_1 + 3x_2)f(x_1, x_2) + (x_1 + 3x_2)u \end{aligned}$$

- If $x_1 + 3x_2 \neq 0$, then it is always possible to choose u so that $\frac{dV}{dt} < 0$.
- If $x_1 + 3x_2 = 0$, then $\frac{dV}{dt} = -3x_2^2 \leq 0$. Moreover if $x_2 = 0$ then $x_1 = 0$, and so we can conclude that $\frac{dV}{dt} < 0$, for $\mathbf{x} \neq \mathbf{0}$.

(b) Again it is straight forward to show that $V(\mathbf{x}) = \frac{1}{2}x_1^2 + x_1x_2 + x_2^2$ is p.d., and

$$\begin{aligned}
\frac{dV}{dt} &= x_1\dot{x}_1 + \dot{x}_1x_2 + x_1\dot{x}_2 + 2x_2\dot{x}_2 \\
&= (x_1 + x_2)\dot{x}_1 + (x_1 + 2x_2)\dot{x}_2 \\
&= \underbrace{(x_1 + x_2)(x_2 + 4x_1^2x_2) - (x_1 + 2x_2)x_1}_{a(\mathbf{x})} + \underbrace{(x_1 + 2x_2)}_{b(\mathbf{x})}u
\end{aligned}$$

- If $x_1 + 2x_2 \neq 0$, then it is always possible to choose u so that $\frac{dV}{dt} < 0$.
- If $x_1 + 2x_2 = 0$, then $\frac{dV}{dt} = -x_2^2 - 16x_2^4 \leq 0$. Moreover if $x_2 = 0$ then $x_1 = 0$, and so we can conclude that $\frac{dV}{dt} < 0$, for $\mathbf{x} \neq \mathbf{0}$.

There are many controls that will stabilise the system. For one, see the notes on *feedback linearisation*. Another is given by *Sontag*'s formula:

$$u(\mathbf{x}) = \begin{cases} -\frac{a+\sqrt{a^2+b^4}}{b}, & \text{if } x_1 + 2x_2 \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

where a and b are as given above.

(c) V is obviously p.d.

$$\begin{aligned}
\triangle V &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\
&= x_2^2 + 2(x_1^2 + u)^2 - x_1^2 - 2x_2^2 \\
&= 2(x_1^2 + u)^2 - x_1^2 - x_2^2
\end{aligned}$$

Choose $u = -x_1^2$ to get $\triangle V < 0$ for $\mathbf{x} \neq \mathbf{0}$.