1. (a) Compute the decoupling index (d_i) for each output (y_i): the smallest nonnegative integer such that $C_i A^{d_i} B \neq \mathbf{0}$, or alternatively repeatedly differentiate y_i with respect to time until the first appearance of $\mathbf{u}$ - then d_i equals the order of the derivative involved less one.

$$\begin{aligned} y_1 &= x_1 \\ \Rightarrow \dot{y}_1 &= \dot{x}_1 = x_2 + \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{u} \qquad \text{implies } d_1 = 0 \end{aligned}$$

$$\begin{aligned} y_2 &= x_3 \\ \Rightarrow \dot{y}_1 &= \dot{x}_3 = -x_3 + \begin{pmatrix} 3 & 7 \end{pmatrix} \mathbf{u} \qquad \text{implies } d_2 = 0 \end{aligned}$$

Then

$$N \triangleq \begin{pmatrix} C_1 A^{d_1} B \\ C_2 A^{d_2} B \end{pmatrix} = CB = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

The system can be decoupled by linear state feedback if and only if N^{-1} exists. Here $det N = 1 \neq 0$ and so the system is decouplable.

(b)

$$\begin{aligned} y_1 &= 3x_1 + 6x_2 + x_3 \\ \Rightarrow \dot{y}_1 &= 3\dot{x}_1 + 6\dot{x}_2 + \dot{x}_3 \\ &= 3(x_2) + 6(x_3) + \left(-6x_1 - 11x_2 - 6x_3 + \begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{u}\right) \qquad \text{implies } d_1 = 0 \end{aligned}$$

$$\begin{aligned} y_2 &= 2x_1 \\ \Rightarrow \dot{y}_2 &= 2\dot{x}_1 = 2(x_2) \\ \Rightarrow \ddot{y}_2 &= 2\dot{x}_2 = 2(x_3) \\ \Rightarrow \dddot{y}_2 &= 2\dot{x}_3 = 2\left(-6x_1 - 11x_2 - 6x_3 + \begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{u}\right) \qquad \text{implies } d_2 = 2 \end{aligned}$$

Thus

$$N = \begin{pmatrix} C_1 A^0 B \\ C_2 A^2 B \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix}$$

and since $det N = 0$, the system is not decouplable.

(c)

$$\begin{aligned} y_1 &= 2x_1 + x_2 \\ \Rightarrow \dot{y}_1 &= 2\dot{x}_1 + \dot{x}_2 \\ &= 2\left(x_2 + \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{u}\right) + \left(4x_1 - 3x_2 + \begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{u}\right) \qquad \text{implies } d_1 = 0 \end{aligned}$$

$$\begin{aligned} y_2 &= x_2 \\ \Rightarrow \dot{y}_2 &= \dot{x}_2 = \left(4x_1 - 3x_2 + \begin{pmatrix} 0 & -3 \end{pmatrix} \mathbf{u}\right) \qquad \text{implies } d_2 = 0 \end{aligned}$$

Thus

$$N = \begin{pmatrix} C_1 A^0 B \\ C_2 A^0 B \end{pmatrix} = CB = \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix}$$

and since $det N = -6 \neq 0$, the system is decouplable.

2. Systems (a) & (c) of Q1 may be decoupled by linear state feedback.

   (a) The required feedback is $\mathbf{u} = N^{-1}(-G\mathbf{x} + \mathbf{v})$, where G is the matrix whose i-th row is $C_i A^{d_i+1}$.
   For system 1-(a) this is

$$G = CA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

   and so

$$\mathbf{u} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \left\{ - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x} + \mathbf{v} \right\} = \begin{pmatrix} 0 & -7 & -2 \\ 0 & 3 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \mathbf{v}.$$

   For system 1-(c) this is

$$G = CA = \begin{pmatrix} 4 & -1 \\ 4 & -3 \end{pmatrix}$$

   and so

$$\mathbf{u} = \begin{pmatrix} 1/2 & 1/6 \\ 0 & -1/3 \end{pmatrix} \left\{ - \begin{pmatrix} 4 & -1 \\ 4 & -3 \end{pmatrix} \mathbf{x} + \mathbf{v} \right\} = \begin{pmatrix} -8/3 & 1 \\ 4/3 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1/2 & 1/6 \\ 0 & -1/3 \end{pmatrix} \mathbf{v}.$$

   (b) The remnant of a decoupled system consists of those modes that are not observable by the output. For decoupling to be useful, the remnant must be stable. After the decoupling control has been applied, the state equation becomes $\dot{\mathbf{x}} = (A - BN^{-1}G)\mathbf{x} + BN^{-1}\mathbf{v}$. The matrix $A - BN^{-1}G$ has $q \triangleq \sum_{i=1}^{p}(d_i + 1)$ eigenvalues at 0. The remaining $n - q$ eigenvalues are those of the remnant.
   For system 1-(a), $q = 2$ and

$$\begin{aligned} A - BN^{-1}G = A + B(-N^{-1}G) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 0 & -7 & -2 \\ 0 & 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & -6 & -2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -8 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

   which has characteristic polynomial $\chi(\lambda) = \lambda^2(\lambda + 8)$. Hence the remnant has characteristic polynomial $\lambda + 8$, i.e. it is stable.
   For system 1-(c), $q = 2 = n$ and so there is no remnant.

   (c) After integrator decoupling, for both input/output channels of 1-(a) and 1-(c), the mathematical description is

$$\frac{dy_i}{dt} = v_i, \qquad i = 1, 2.$$

   For the purposes of completing the control design we consider each of these to be given O-L systems. Take any one of these SISO subsystems. It is first order and the natural choice for "state variable" is $x = y_i$. The resultant state description is $\dot{x} = [0]x + [1]v$ with O-L characteristic polynomial λ. The system is in CCF: to place the C-L pole at -2 (i.e. the desired C-L characteristic polynomial is $\lambda + 2$), the feedback matrix is $K = [k_1] = [a_0 - \hat{a}_0] = [-2]$, and the feedback control law is $v = -2x + w$ where w is some new reference input. Translating this back into a description for each of the channels gives

$$v_i = -2y_i + w_i, \qquad i = 1, 2$$

or in terms of the state space description of the **original** system

$$v_i = -2C_i\mathbf{x} + w_i, \qquad i = 1, 2$$

or

$$\mathbf{v} = -2C\mathbf{x} + \mathbf{w}$$

3. Consider the expression for the i-th output

$$y_i = C_i\mathbf{x} + D_i\mathbf{u}$$

where C_i and D_i are the i-th rows of C and D respectively. If $D_i = \mathbf{0}$ then the decoupling index d_i is defined in exactly the same way as before. If however $D_i \neq \mathbf{0}$, then $d_i \triangleq -1$; i-row of N is set equal to D_i and the i-th row is then C_i. Otherwise, N and G are calculated as before.
(The A, B, and C matrices of the given system are the same as those of Q1-(a); see there for any relevant calculations) We see that $D_1 \neq \mathbf{0}$, so $d_1 = -1$. As before $d_2 = 0$.

$$N = \begin{pmatrix} D_1 \\ C_2A^0B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 7 \end{pmatrix}$$

for which $detN \neq 0$; so the system may be decoupled by state feedback. In addition

$$G = \begin{pmatrix} C_1 \\ C_2A \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The decoupling control is

$$\begin{aligned} \mathbf{u} = N^{-1}(-G\mathbf{x} + \mathbf{v}) &= \begin{pmatrix} 7/4 & -1/4 \\ -3/4 & 1/4 \end{pmatrix} \left\{ -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x} + \mathbf{v} \right\} \\ &= \begin{pmatrix} -7/4 & 0 & -1/4 \\ 3/4 & 0 & 1/4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 7/4 & -1/4 \\ -3/4 & 1/4 \end{pmatrix} \mathbf{v}. \end{aligned}$$

Here $q = 1$ and thus the remnant has order 2.

$$\begin{aligned} A - BN^{-1}G = A + B(-N^{-1}G) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} -7/4 & 0 & -1/4 \\ 3/4 & 0 & 1/4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} -1/4 & 0 & 1/4 \\ -3/2 & 0 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1/4 & 1 & 1/4 \\ -3/2 & -2 & 5/2 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

which has characteristic polynomial $\lambda(\lambda^2 + (9/4)\lambda + 2)$. Hence the remnant has characteristic polynomial $\lambda^2 + (9/4)\lambda + 2$, which is readily checked to be stable.