1. (See Problem Sheet 3 Q 2(a)). The system is CC.
   A has characteristic polynomial $\lambda^2 - 3\lambda + 2$ (Eigenvalues or O-L poles are $1, 2$). The CCF description is therefore

$$\dot{\hat{\mathbf{x}}} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \hat{\mathbf{x}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

obtained by the transformation is $\mathbf{x} = \hat{T}\hat{\mathbf{x}}$ where

$$\hat{T} = U\hat{U}^{-1} = \begin{pmatrix} 3 & 11 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

Let's place the C-L poles at $-1, -2$ say. The desired characteristic equation is then $\lambda^2 + 3\lambda + 2$. The feedback matrix for the transformed system is

$$\hat{K} = \begin{pmatrix} \hat{k}_1 & \hat{k}_2 \end{pmatrix}$$

where

$$\begin{aligned} \hat{k}_1 &= a_0 - \hat{a}_0 = 2 - 2 = 0 \\ \hat{k}_2 &= a_1 - \hat{a}_1 = -3 - 3 = -6 \end{aligned}$$

and the feedback matrix for the original system may be computed by

$$K = \hat{K}\hat{T}^{-1} = \begin{pmatrix} 0 & -6 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -12 \end{pmatrix}$$

The linear state feedback controller is described by

$$u = K\mathbf{x} + v$$

and the closed loop system is

$$\dot{\mathbf{x}} = (A + BK)\mathbf{x} + Bv = \begin{pmatrix} -5 & 4 \\ -3 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} v$$

2. (See Problem Sheet 3 Q 2(c)). The system is CC.
   A has characteristic polynomial $\lambda^3 + 6\lambda^2 + 11\lambda + 6$ (Eigenvalues or O-L poles are $-1, -2$ and -3). The CCF description is therefore

$$\dot{\hat{\mathbf{x}}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} \hat{\mathbf{x}} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

obtained by the transformation is $\mathbf{x} = \hat{T}\hat{\mathbf{x}}$ where

$$\hat{T} = U\hat{U}^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -11 \end{pmatrix} \begin{pmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -10 & -6 & -1 \\ 6 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix}$$

Let's place the C-L poles at $-4 \pm i, -6$ say. The desired characteristic equation is then $\lambda^3 + 14\lambda^2 + 65\lambda + 102$. The feedback matrix for the transformed system is

$$\hat{K} = \begin{pmatrix} \hat{k}_1 & \hat{k}_2 & \hat{k}_3 \end{pmatrix}$$

where

$$\begin{aligned} \hat{k}_1 &= a_0 - \hat{a}_0 = 6 - 102 = -96 \\ \hat{k}_2 &= a_1 - \hat{a}_1 = 11 - 65 = -54 \\ \hat{k}_3 &= a_2 - \hat{a}_2 = 6 - 14 = -8 \end{aligned}$$

and the feedback matrix for the original system may be computed by

$$K = \hat{K}\hat{T}^{-1} = \begin{pmatrix} -96 & -54 & -8 \end{pmatrix} \begin{pmatrix} -0.1 & 0 & -0.1 \\ 0.6 & 1.0 & 0.6 \\ -3.6 & -6.0 & -2.6 \end{pmatrix} = \begin{pmatrix} 6 & -6 & -2 \end{pmatrix}$$

The linear state feedback controller is described by

$$u = K\mathbf{x} + v$$

and the closed loop system is

$$\dot{\mathbf{x}} = (A + BK)\mathbf{x} + Bv = \begin{pmatrix} -6 & 7 & 2 \\ 0 & 0 & 1 \\ 0 & -17 & -8 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} v$$

3. For this system

$$V = \begin{pmatrix} c \\ cA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -5 & 4 \end{pmatrix}$$

which has rank $= 2$. Hence system is CO.
The system has characteristic polynomial $\lambda^2+5\lambda+4 = (\lambda+4)(\lambda+1)$. Hence we place the observer poles at $-12, -12$ which gives a observer characteristic polynomial of $\lambda^2 + 24\lambda + 144$. To design the observer we apply the pole placement algorithm to the dual system. The dual system is

$$\dot{\mathbf{x}}^* = \begin{pmatrix} -5 & -1 \\ 4 & 0 \end{pmatrix} \mathbf{x}^* + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u^*$$

The transformed dual system is

$$\dot{\tilde{\mathbf{x}}}^* = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix} \tilde{\mathbf{x}}^* + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u^*$$

obtained by the transformation $\mathbf{x}^* = \hat{T}\tilde{\mathbf{x}}$ where

$$\hat{T} = U^*\tilde{U}^{-1} = \begin{pmatrix} 1 & -5 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

The "feedback" matrix for the transformed dual system is

$$\hat{K}^* = \begin{pmatrix} \hat{k}_1^* & \hat{k}_2^* \end{pmatrix}$$

where

$$\begin{aligned}\hat{k}_1^* &= a_0 - \hat{a}_0 = 4 - 144 = -140 \\ \hat{k}_2^* &= a_1 - \hat{a}_1 = 5 - 24 = -19\end{aligned}$$

and the feedback matrix for the original dual system may be computed by

$$K^* = \hat{K}\hat{T}^{-1} = \begin{pmatrix} -140 & -19 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -19 & -35 \end{pmatrix}$$

Back in the original system L is chosen to be $(K^*)^T$. Hence the observer design is

$$\begin{aligned}\dot{\mathbf{z}} &= (A + LC)\mathbf{z} - Ly \\ &= \begin{pmatrix} -24 & 4 \\ -36 & 0 \end{pmatrix} \mathbf{z} - \begin{pmatrix} -19 \\ -35 \end{pmatrix} y\end{aligned}$$

4. For the system of Q1, we have

$$V = \begin{pmatrix} c \\ cA \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 7 & -12 \end{pmatrix}$$

which has rank = 2. Hence system is CO.

The feedback design of Q1 placed the C-L system poles at $-1, -2$. Using the "factor of 3" rule-of-thumb for observer design, we'll place the observer poles at $-6, -6$ which gives a observer characteristic polynomial of $\lambda^2 + 12\lambda + 36$. Again, to design the observer we'll apply the pole placement algorithm to the dual system. The dual system is

$$\dot{\mathbf{x}}^* = \begin{pmatrix} -23 & -15 \\ 40 & 26 \end{pmatrix} \mathbf{x}^* + \begin{pmatrix} 1 \\ -2 \end{pmatrix} u^*$$

The transformed dual system is

$$\dot{\tilde{\mathbf{x}}}^* = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \tilde{\mathbf{x}}^* + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u^*$$

obtained by the transformation $\mathbf{x}^* = \hat{T}\tilde{\mathbf{x}}$ where

$$\hat{T} = U^*\tilde{U}^{-1} = \begin{pmatrix} 1 & 7 \\ -2 & -12 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -6 & -2 \end{pmatrix}$$

The "feedback" matrix for the transformed dual system is

$$\hat{K}^* = \begin{pmatrix} \hat{k}_1^* & \hat{k}_2^* \end{pmatrix}$$

where

$$\begin{aligned}\hat{k}_1^* &= a_0 - \hat{a}_0 = 2 - 36 = -34 \\ \hat{k}_2^* &= a_1 - \hat{a}_1 = -3 - 12 = -15\end{aligned}$$

and the feedback matrix for the original dual system may be computed by

$$K^* = \hat{K}\hat{T}^{-1} = \begin{pmatrix} -34 & -15 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 11 & 13 \end{pmatrix}$$

Back in the original system L is chosen to be $(K^*)^T$. Hence the observer design is

$$\begin{aligned}\dot{\mathbf{z}} &= (A + LC)\mathbf{z} - Ly + Bu \\ &= \begin{pmatrix} -12 & 18 \\ -2 & 0 \end{pmatrix} \mathbf{z} - \begin{pmatrix} 11 \\ 13 \end{pmatrix} y + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u\end{aligned}$$

In summary, the complete specification is

(a) (System)

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{pmatrix} -23 & 40 \\ -15 & 26 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x}\end{aligned}$$

(b) (State Estimator)

$$\dot{\mathbf{z}} = \begin{pmatrix} -12 & 18 \\ -2 & 0 \end{pmatrix} \mathbf{z} - \begin{pmatrix} 11 \\ 13 \end{pmatrix} y + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u$$

(c) (Feedback control)

$$u = \begin{pmatrix} 6 & -12 \end{pmatrix} \mathbf{z} + v$$