1. Recall from Linear Algebra that

   - a necessary and sufficient condition for diagonalisability of $n \times n$ matrix is that n linearly independent eigenvectors can be found, i.e. E is invertible
   - a sufficient condition is that the eigenvalues be distinct.

   The matrices A_1 and A_3 have distinct eigenvalues (-1, -3 and $\pm 2i$ respectively) and so are diagonalisable. Note that A_2 also has distinct eigenvalues, but is already in diagonal form.
   A_3 has the repeated eigenvalue -2, but only eigenvectors which are multiples of $(1, -2)^T$, and so is not diagonalisable. A_4 and A_6 similarly have repeated eigenvalues - check the diagonal entries, and both have only 2 linearly independent eigenvectors - $\{(1, 0, 0)^T,\ (0, 0, 1)^T\}$ and $\{(1, 0, 0)^T,\ (2, 2, 1)^T\}$ respectively; hence neither is diagonalisable.
   A_2, A_4 and A_6 are in Jordan form.

$$A_1^k = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} (-1)^k & 0 \\ 0 & (-3)^k \end{pmatrix} \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$e^{A_1 t} = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$A_2^k = \begin{pmatrix} 2^k & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (\frac{1}{2})^k \end{pmatrix}$$

$$A_3^k = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} (-2)^k & k(-2)^{k-1} \\ 0 & (-2)^k \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$A_4^k = \begin{pmatrix} 2^k & k2^{k-1} & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & (-1)^k \end{pmatrix}$$

$$e^{A_4 t} = \begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

$$\begin{aligned} A_5^k &= \begin{pmatrix} 1 & 1 \\ 2i & -2i \end{pmatrix} \begin{pmatrix} (2i)^k & 0 \\ 0 & (-2i)^k \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{i}{4} \\ \frac{1}{2} & \frac{i}{4} \end{pmatrix} \\ &= \begin{cases} (-4)^{k/2} I, & \text{if } k \text{ is even} \\ (-4)^{(k-1)/2} A_5, & \text{if } k \text{ is odd} \end{cases} \end{aligned}$$

Using the last expression for A_5^k, it is possible to show that

$$e^{A_5 t} = \cos 2tI + \frac{\sin 2t}{2} A_5 = \begin{pmatrix} \cos 2t & \frac{\sin 2t}{2} \\ -2\sin 2t & \cos 2t \end{pmatrix}$$

and finally

$$A_6^k = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (-1)^k & k(-1)^{k-1} & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & 0^k \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$e^{A_6 t} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

2. (a) The solution of this ode is

$$\begin{aligned} \mathbf{x}(t) &= e^{A_1 t}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \int_0^t e^{A_1(t-\tau)}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\left(1 - e^{-2\tau}\right) d\tau \\ &= \int_0^t \begin{pmatrix} \frac{1}{2}e^{-(t-\tau)} - \frac{1}{2}e^{-3(t-\tau)} \\ -\frac{1}{2}e^{-(t-\tau)} + \frac{3}{2}e^{-3(t-\tau)} \end{pmatrix}\left(1 - e^{-2\tau}\right) d\tau \\ &= \begin{pmatrix} -\frac{1}{3}e^{-3t} - e^{-t} + \frac{1}{3} + e^{-2t} \\ e^{-3t} + e^{-t} - 2e^{-2t} \end{pmatrix} \end{aligned}$$

(b) The solution of this difference equation is

$$\begin{aligned} \mathbf{x}_k &= A_3^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{j=0}^{k-1} A_3^{k-1-j}\begin{pmatrix} 0 \\ 1 \end{pmatrix} 1. \\ &= \begin{pmatrix} \frac{(8-12k)(-2)^k+1}{9} \\ \frac{(8+24k)(-2)^k+1}{9} \end{pmatrix} \end{aligned}$$

(c) The solution of this ode is

$$\begin{aligned} \mathbf{x}(t) &= e^{A_5 t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \int_0^t e^{A_5(t-\tau)}\begin{pmatrix} 0 \\ 1 \end{pmatrix}(3\tau)\, d\tau \\ &= \begin{pmatrix} -\frac{7}{8}\sin 2t + \cos 2t + \frac{3}{4}t \\ -\frac{7}{4}\cos 2t - 2\sin 2t + \frac{3}{4} \end{pmatrix} \end{aligned}$$