UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4328 SEMESTER: Spring 2015

MODULE TITLE: Mathematical Control Theory DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke PERCENTAGE OF TOTAL MARKS: 80 %

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions. There are notes & formulae supplied.**

Unless otherwise stated, the following notation is used throughout: $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{y}$ are the $n \times 1$ state, $r \times 1$ input and $m \times 1$ output vectors respectively, whose components are real functions of time. $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}_i(\mathbf{x}),\ i = 1, 2, \dots r$ are $n \times 1$ C^1 vector fields. A is a $n \times n$ state matrix, B a $n \times r$ input matrix and C a $m \times n$ output matrix, all of whose elements are real numbers. LTI is an acronym for linear time invariant.

1. (a) Define complete controllability and complete observability for the LTI continuous-time system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x}.$$

3

(b) By using the *Popov-Belevitch-Hautus* (PBH) tests determine whether the system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$$

is completely controllable and/or completely observable. 7

(c) State the *Kalman* canonical structure theorem. Classify the modes of the system of part (b) according to the theorem. 3

(d) Define stabilisability and detectability. Is the system of part (b) stabilisable, detectable ? 3

2. (a) State *La Salle*'s Invariance Principle and discuss how it may be used to establish asymptotic stability of the origin for the continuous-time system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}).$$

3

(b) Using the Invariance Principle or otherwise, determine whether the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2^3 - c(x_1)$$

where

$$c(x_1) = \begin{cases} -1 + e^{x_1}, & \text{if } x_1 \le 0, \\ 1 - e^{-x_1}, & \text{if } 0 < x_1 \end{cases}$$

is an asymptotically stable fixed point. Is it globally asymptotically stable? 5

(c) Can the control u be chosen to globally stabilise the origin of the uncertain system

$$\ddot{x} + \beta \dot{x}^2 + \alpha \sinh x = u$$

where $|\beta| < 2$ and $1 < \alpha < 2$? If so, suggest a possible control; if not, prove it. 8

3. (a) State the pole placement problem for the LTI continuous-time system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

with linear state feedback

$$\mathbf{u} = K\mathbf{x} + \mathbf{v}$$

and give a sufficient condition under which it can be solved. 4

(b) Find a linear state feedback controller that places the unstable pole(s) of the open loop system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

at $\lambda = -1$ in the complex plane. 8

(c) If the complete state variable is not available for measurement but only the output $y = x_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \mathbf{x}$, prove that is it possible to design a *Luenberger* observer to estimate the state variables. What is the minimum possible order of such an observer ? 4

4. (a) For the affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{r} \mathbf{g}_i(\mathbf{x})u_i, \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}),$$

define what is meant by a *control-Lyapunov function*. 3

(b) For what range of values of α is

$$V(\mathbf{x}) = x_1^2 + 2\alpha x_1 x_2 + x_2^2$$

a control-*Lyapunov* function for the system

$$\dot{\mathbf{x}} = \begin{pmatrix} x_2 \\ x_1 \cosh x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

10

(c) State *Artstein*'s Theorem, and use the equivalent *Sontag*'s formula to find a stabilising control for the system and *control-Lyapunov function* of part (b) using an admissible value for α. 3

5. For the nonlinear continuous-time system

$$\dot{x} = x\cos x + u$$

   (a) by using the *Hamilton-Jacobi-Bellman* equation, show that the stabilising feedback control that minimises the cost functional

$$J(u) = \int_0^\infty x^2 + \frac{1}{3}u^2\, dt$$

   for the system is given by

$$u = u^* \triangleq -x\cos x - x\sqrt{\cos^2 x + 3}.$$

8

   (b) For the system, design a control by feedback linearisation (cancellation of nonlinearities) which transforms the system to the linear system

$$\dot{x} = -kx$$

   and then places the closed-loop pole of this linear system at $\lambda = -2$. 4

   (c) Comment on the designs and how they are related. 4

6. The controlled *Rössler* prototype-4 system of a chemical reaction with three reactants of concentrations $x_1,\ x_2$ and x_3 respectively is described by

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3\\ \dot{x}_2 &= x_1\\ \dot{x}_3 &= a(x_2 - x_2^2) - bx_3 + u\end{aligned}$$

where a and b are positive parameters and u is the scalar control.

   (a) For the single input-single output affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \qquad y = h(\mathbf{x})$$

   define the relative degree at $\mathbf{x} = \mathbf{x}_0$. What is a global relative degree ? 3

   (b) Show that the *Rössler* system has a global relative degree of 3 if $y = x_2$ and hence find the corresponding state transformation and linearising control that transforms the system to *normal form*. 8

   (c) If $a = 0.5 = b$ find the feedback control that places the poles of the exact linearised system at $-1, -1, -1$. 5