UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4328

SEMESTER: Spring 2014

MODULE TITLE: Mathematical Control Theory

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 80 %

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions. There are notes & formulae supplied.**

Unless otherwise stated, the following notation is used throughout: $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{y}$ are the $n \times 1$ state, $r \times 1$ input and $m \times 1$ output vectors respectively, whose components are real functions of time. $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}_i(\mathbf{x}),\ i = 1, 2, \ldots r$ are $n \times 1$ C^1 vector fields. A is a $n \times n$ state matrix, B a $n \times r$ input matrix and C a $m \times n$ output matrix, all of whose elements are real numbers. LTI is an acronym for linear time invariant.

1. (a) Define complete controllability and complete observability for the LTI continuous-time system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x}.$$

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(b) By using the *Popov-Belevitch-Hautus* (PBH) tests determine whether the system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 3 & -1 \end{pmatrix}$$

is completely controllable and/or completely observable. 7

(c) State the *Kalman* canonical structure theorem. Classify the modes of the system of part (b) according to the theorem. 3

(d) Define stabilisability and detectability. Is the system of part (b) stabilisable, detectable ? 3

2. (a) State *La Salle*'s Invariance Principle and discuss how it may be used to establish asymptotic stability of the origin for the continuous-time system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}).$$

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(b) Using the Invariance Principle or otherwise, determine whether the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -b(x_2) - x_1 \cosh x_1$$

where

$$b(x_2) = \begin{cases} \frac{x_2}{\sqrt{|x_2|}}, & \text{if } x_2 \neq 0, \\ 0, & \text{if } x_2 = 0 \end{cases}$$

is an asymptotically stable fixed point. Is it globally asymptotically stable? 5

(c) Can the control u be chosen to stabilise the uncertain system

$$\ddot{x} + \beta\dot{x}^3 + \alpha|x| = u$$

where $-2 < \beta < 1$ and $-1 < \alpha < 2$? If so, suggest a possible control; if not, prove it. 8

3. (a) State the pole placement problem for the LTI continuous-time system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

with linear state feedback

$$\mathbf{u} = K\mathbf{x} + \mathbf{v}$$

and give a sufficient condition under which it can be solved. 4

(b) Find a linear state feedback controller that places the unstable pole(s) of the open loop system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

at $\lambda = -1$ in the complex plane. 8

(c) If the complete state variable is not available for measurement but only the output $y = x_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{x}$, prove that is it possible to design a *Luenberger* observer to estimate the state variables. What is the minimum possible order of such an observer ? 4

4. (a) For the affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{r} \mathbf{g}_i(\mathbf{x})u_i, \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}),$$

define what is meant by a *control-Lyapunov function*. 3

(b) For what range of values of α is

$$V(\mathbf{x}) = \frac{2}{3}x_1^2 + 2\alpha x_1 x_2 + x_2^2$$

a control-*Lyapunov* function for the system

$$\dot{\mathbf{x}} = \begin{pmatrix} \cos x_2 \\ x_1 + x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u.$$

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(c) State *Artstein*'s Theorem, and use the equivalent *Sontag*'s formula to find a stabilising control for the system and *control-Lyapunov function* of part (b) using an admissible value for α. 3

5. For the nonlinear continuous-time system

$$\dot{x} = \sin x + u$$

   (a) by using the *Hamilton-Jacobi-Bellman* equation, show that the stabilising feedback control that minimises the cost functional

$$J(u) = \int_0^\infty x^2 + xu + u^2 \, dt$$

   for the system is given by

$$u = u^* \triangleq -\sin x - x\sqrt{\left(\frac{1}{2} - \frac{\sin x}{x}\right)^2 + \frac{3}{4}}.$$

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   (b) For the system design a control by feedback linearisation (cancellation of nonlinearities) which transforms the system to the linear system

$$\dot{z} = v$$

   and then places the closed-loop pole of this linear system at $\lambda = -1$. 5

   (c) Comment on the designs and how they are related. 3

6. A highly infectious disease whose effects can be controlled by vaccination is modelled by

$$\frac{dS}{dt} = -aSI - U, \qquad \frac{dI}{dt} = aSI - rI$$

where $S(t)$ and $I(t)$ are the normalised numbers of susceptible and infected people at time t respectively; $a > 0$ and $r > 0$ are the infection rate and recovery rate respectively, and the control input $U(t)$ is the vaccination rate.

   (a) Find the feedback linearisation scheme which transforms this affine control system to

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v.$$

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   (b) Find the control U that will result in the linear transformed system of part (a) having the closed-loop characteristic equation

$$\lambda^2 + 2a\lambda + 2r = 0.$$

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