UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering
Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4328

SEMESTER: Spring 2012

MODULE TITLE: Mathematical Control Theory

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke

PERCENTAGE OF TOTAL MARKS: 100 %

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Full marks for correct answers to 5 questions. There are notes & formulae supplied.**

Unless otherwise stated, the following notation is used throughout: $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{y}$ are the $n \times 1$ state, $r \times 1$ input and $m \times 1$ output vectors respectively, whose components are real functions of time. $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}_i(\mathbf{x}),\ i = 1, 2, \ldots r$ are $n \times 1$ C^1 vector fields. A is a $n \times n$ state matrix, B a $n \times r$ input matrix and C a $m \times n$ output matrix, all of whose elements are real numbers. LTI is an acronym for linear time invariant.

1. (a) Define complete controllability and complete observability for the LTI discrete-time system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \qquad \mathbf{y}_k = C\mathbf{x}_k.$$

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(b) By using the modal transformation determine whether the system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 2 & 1 \end{pmatrix}.$$

is completely controllable and/or completely observable. 8

(c) State the *Kalman* canonical structure theorem. Classify the modes of the system of part (b) according to the theorem. 4

(d) Define stabilisability and detectability. Is the system of part (b) stabilisable, detectable ? 4

2. (a) Describe *Lyapunov*'s 2nd or Direct Method, clearly stating the relevant theorems for the continuous-time system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}).$$

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(b) State *La Salle*'s Invariance Principle and discuss how it may be used to determine asymptotic stability. 3

(c) Determine whether the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2^3 - x_1\sqrt{|x_1|}$$

is an asymptotically stable fixed point. Is it globally asymptotically stable? 6

(d) Can the control u be chosen to stabilise the system

$$\ddot{x} + \beta\dot{x}^2 + \alpha x = u$$

where $|\beta| < 2$ and $-4 < \alpha < -2$? If so, suggest a possible control; if not, prove it. 8

3. (a) State the pole placement problem for the LTI continuous-time system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

with linear state feedback

$$\mathbf{u} = K\mathbf{x} + \mathbf{v}.$$

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(b) Derive an algorithm for placing the poles of a completely controllable single input system using linear state feedback at specified locations in the complex plane. 6

(c) Hence find a linear state feedback controller that places the unstable pole(s) of the open loop system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

at $\lambda = -1$ in the complex plane. 8

(d) If the complete state variable is not available for measurement but only the output $y = x_3 = \begin{pmatrix} 0, & 0, & 1 \end{pmatrix} \mathbf{x}$, is it possible to design a *Luenberger* observer to estimate the state variables? 4

4. (a) A square system is one in which the number of inputs equals the number of outputs. For the square LTI discrete-time system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \qquad \mathbf{y}_k = C\mathbf{x}_k,$$

define the decoupling indices and derive the conditions under which the system may be "delay-decoupled" by linear state feedback

$$\mathbf{u}_k = K\mathbf{x}_k + M\mathbf{v}_k$$

6

(b) Can the system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 \\ 0 & 1 \\ 1 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

be delay-decoupled ? 8

(c) What is the remnant of an delay-decoupled system, and why is it important ? Is delay-decoupling an appropriate design tool for the system of part (b) ? 6

5. (a) For the affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{r} \mathbf{g}_i(\mathbf{x})u_i, \qquad \mathbf{0} = \mathbf{f}(\mathbf{0}),$$

define what is meant by a *control-Lyapunov function*. 6

(b) For what range of values of α is

$$V(\mathbf{x}) = \frac{1}{2}x_1^2 + \alpha x_1 x_2 + \frac{1}{2}x_2^2$$

a control-*Lyapunov* function for the system

$$\dot{\mathbf{x}} = \begin{pmatrix} x_2 \\ F(x_1, x_2) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

where $F(0,0) = 0$. 14

6. This question considers three design approaches to the control of the non-linear continuous-time system

$$\dot{x} = \sin x + 2xu.$$

(a) Using the *Hamilton-Jacobi-Bellman* equation, show that the stabilising feedback control that minimises the cost functional

$$J(u) = \int_0^\infty \left(x^2 + u^2\right)\, dt$$

for the system is given by

$$u = u^* \triangleq -\frac{\sin x}{2x} - \sqrt{\left(\frac{\sin x}{2x}\right)^2 + x^2}.$$

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(b) For the system design a control by feedback linearisation (cancellation of nonlinearities) which (i) transforms the system to the linear system

$$\dot{z} = v$$

and (ii) then places the pole of this linear system at $\lambda = -2$. 4

(c) By solving the *Lyapunov* equation for the linear system of part (b) [after feedback has been applied], find a quadratic *Lyapunov* function which can be used as a control- *Lyapunov* function for the original system. Derive a control based on *Sontag*'s formula which will stabilise the system. 4

(d) Comment on the designs and how they are related. 3

7. (a) State the conditions under which the single input affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \qquad \mathbf{0} = \mathbf{f}(\mathbf{0})$$

can be transformed by *feedback linearisation* to

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \mathbf{z} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v$$

and give an algorithm for doing so. 4

(b) Find the feedback linearisation scheme which transforms the system

$$\dot{\mathbf{x}} = \begin{pmatrix} x_2 \\ -x_1 - x_2^2 \end{pmatrix} + \begin{pmatrix} x_1 \\ 0 \end{pmatrix} u$$

to the linear form of part (a). 10

(c) Find the control u that will result in the linear transformed system of part (b) having the characteristic equation $\lambda^2 + 2\lambda + 1 = 0$. 6

8. (a) Describe the stabilisation problem for the switched LTI system

$$\dot{\mathbf{x}} = A_\sigma \mathbf{x}, \qquad \sigma \in \{1, 2, \ldots, p\}.$$

3

(b) By analysing the eigenstructure of the individual components of the switched linear system with $p = 2$ and

$$A_1 = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & -4 \\ -2 & 1 \end{pmatrix},$$

determine whether the system is stabilisable. 12

Determine a switching strategy which drives $(1,1)^T$ to $\mathbf{0}$. 5