UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4315

SEMESTER: Spring 2012

MODULE TITLE: Operations Research 2

DURATION OF EXAMINATION: 2 1/2 hours

LECTURER: Dr. M. Burke/Dr. J. Kinsella

GRADING SCHEME:

EXTERNAL EXAMINER: Prof. T. Myers

INSTRUCTIONS TO CANDIDATES: **Answer four questions correctly for full marks. Answer two questions from Q.1–Q.3 and two questions from Q.4–Q.6**

1 (a) (i) Explain briefly why the following strategy for the solution of I.P.'s is not useful: "Solve the L.P. relaxation then round off the components of the solution to the nearest integers". 1%

(ii) Explain one method whereby a lower bound for the Standard Form I.P. $z = \max\{c^T x | Ax \le b, x \ge 0, x \in \mathbb{Z}^n\}$ may be found. 1%

(iii) Illustrate your answer by reference to the Travelling Salesman Problem. (N.B. TSP is a min problem!) 1%

(iv) Is the method in your answer easily implemented for any Standard Form I.P.? Explain briefly. 1%

(v) Explain carefully how L.P. Relaxation may be used to find an upper bound for z in a Standard Form I.P.. 2%

(vi) Given an LP (the **Primal** problem) we can write a closely related LP, its **Dual**:

$$\begin{aligned} z &= \max\{c^T x : Ax \le b, x \in \mathbb{R}^n, x \ge 0\} \quad \textbf{Primal} \\ w &= \min\{b^T y : A^T y \ge c, y \in \mathbb{R}^m, y \ge 0\}. \quad \textbf{Dual} \end{aligned}$$

Prove the Weak Duality Theorem: for **any** primal feasible point x and **any** dual feasible point y, $b^T y \ge c^T x$. 3%

(vii) Show that the Weak Duality Theorem implies that **any** feasible point u for the Dual problem gives an upper bound for the optimal solution z of the original IP, namely $z \le b^T u$. 3%

(b) The following Binary Knapsack Problem has the variables ordered so that the "payoff/cost" ratio is decreasing: $\frac{c_j}{a_j} \ge \frac{c_{j-1}}{a_{j-1}}$ for $j = 2, \dots, n$.

$$\begin{aligned} \max 13x_1 + 10x_2 + 15x_3 + 11x_4 + 6x_5 + 2x_6 + 2x_7 \\ 4x_1 + 4x_2 + 7x_3 + 6x_4 + 4x_5 + 2x_6 + 3x_7 \le 12 \\ x \in \mathcal{B}^7 \end{aligned}$$

(i) Use a Greedy Search to find a "good" feasible point. 4%

(ii) Does this give a lower or upper bound for z? Determine which (if either) and if appropriate calculate the bound. 1%

(c) Uncapacitated Facility Location (UFL) problems take the general form:

- Given a set of potential depots $N = \{1, \ldots, n\}$ and a set $M = \{1, \ldots, m\}$ of clients, we need to decide which depots to open and how to utilize them to serve clients.
- The cost c_{ij} of supplying the entire demand of client i from depot j and the cost f_j of building depot j are given.

Given $m = 6$ clients and $n = 4$ depots and costs given by:

$$c_{ij} = \begin{bmatrix} 24 & 77 & 25 & 16 \\ 65 & 9 & 94 & 9 \\ 86 & 31 & 12 & 50 \\ 20 & 50 & 2 & 22 \\ 61 & 80 & 50 & 45 \\ 48 & 43 & 10 & 81 \end{bmatrix} \quad \text{and } f_j = (11, 13, 21, 9).$$

perform **one** iteration of a Local Search to find an improved point (choice of depots to meet demand) given a starting choice. 8%

Start with $S^0 = \{1, 2\}$, so that $c(S^0) = (24 + 9 + 31 + 20 + 61 + 43) + 11 + 13 = 212$.

Show your working clearly.

Hints: If $N = \{1, 2, 3, 4\}$ denotes the set of depots and $S \subseteq N$ denotes the set of open (in use) depots then the cost is:

$$c(S) = \sum_{i=1}^{6} \min_{j \in S} c_{ij} + \sum_{j \in S} f_j.$$

Consider as neighbours all sets obtained from S by the addition or removal of a single element.

2 Consider the Integer Linear Program (IP):

$$\begin{aligned} \max 5x_1 + 4x_2 & \\ x_1 + x_2 & \leq 5 \\ 10x_1 + 6x_2 & \leq 45 \\ x_1, x_2 \leq 0 \text{ and integer.} & \end{aligned}$$

The corresponding Simplex Tableau (transforming the max problem into a min problem) is:

0	-5	-4	0	0
5	1	1	1	0
45	10	6	0	1

N.B. The Simplex Method and the Dual Simplex Method are stated on the last page of this paper.

(a) Apply one iteration of the Simplex Method and show that the Simplex Tableau now takes the form: 6%

22.5	0	-1	0	0.5
0.5	0	0.4	1	-0.1
4.5	1	0.6	0	0.1

(b) After a second iteration of the Simplex Method the Simplex Tableau now takes the form: (**N.B.do not perform the arithmetic!**)

23.75	0	0	2.5	0.25
1.25	0	1	2.5	-0.25
3.75	1	0	-1.5	0.25

Explain why this Tableau is optimal. 1%

(c) The solution to the LP Relaxation of the IP is $x_1 = 3.75, x_2 = 1.25$.
Suppose that we decide to branch on x_1. Explain why the two branches are $S_0 : x_1 \leq 3$ and $S_1 : x_1 \geq 4$. 1%

(d) Consider the branch $S_0 : x_1 \leq 3$.

(i) First show that the basic variable x_1 may be expressed in terms of the non-basic variables x_3 & x_4 as: $x_1 = 3.75 + 1.5x_3 - 0.25x_4$. 2%

(ii) Substitute this expression for x_1 into the S_0 branch constraint and show that it takes the form $1.5x_3 - 0.25x_4 + s = -0.75$. (The variable s is the slack variable for the constraint $x_1 \leq 3$.) 1%

(iii) Show that the Simplex Tableau with the addition of this constraint takes the form: 1%

23.75	0	0	2.5	0.25	0
1.25	0	1	2.5	-0.25	0
3.75	1	0	-1.5	0.25	0
-0.75	0	0	1.5	-0.25	1

(e) Explain why this tableau is optimal but infeasible. 1%

(f) Apply **one** iteration of the Dual Simplex Method to this tableau and show that the Simplex Tableau now takes the form: 6%

23	0	0	4	0	1
2	0	1	1	0	-1
3	1	0	0	0	1
3	0	0	-6	1	-4

(g) This tableau is LP optimal and integer feasible. Explain why. What is the solution to the IP? 2%

(h) Finally, **suppose** that we had started with the branch $S_1 : x_1 \geq 4$, expressed x_1 in terms of the non-basic variables x_3 & x_4 as $x_1 = 3.75 + 1.5x_3 - 0.25x_4$ as above and applied the Dual Simplex method to the resulting tableau.

We would have found (**N.B.do not perform the arithmetic!**)

23.33	0	0	0	0.67	1.67
0.83	0	1	0	0.17	1.67
4	1	0	0	0	-1
0.17	0	0	1	-0.17	-0.67

(i) Is this tableau optimal? 1%

(ii) Is the solution integer? 1%

(iii) What would be the next branch & bound step? (**N.B.do not perform the arithmetic!**) 2%

3 Consider the following problem: a car manufacturer needs to schedule its production for the next four months. To meet its deadlines, the company must produce cars as indicated in column 2 of Table 1.

Production quantities must be integer multiples of 5.

Columns 3 & 4 of the Table give the maximum number of cars that can be produced in each month and the per car production cost in that month.

Cars produced in one month and stored to meet demand in a later month incur a storage cost of €20 per car left in stock at the end of that month.

Month	Cars Needed (D_n)	Max Production (P_n)	Per Car Production Cost (c_n)
1	15	30	€1400
2	20	40	€1300
3	30	35	€1100
4	25	15	€1400

Table 1: Car Production Data

(a) Formulate the problem as a Dynamic Program; specifically identify

(i) the **states** of the system, 1%

(ii) the **stages/iterations**, 1%

(iii) Given that $f_n(s)$ is the minimum cost starting from state s at iteration n and that $f_n(s, x_n)$ is the minimum cost starting from state s at iteration n and choosing x_n; give the relationship (recurrence formula) linking $f_n(s, x_n)$ to f_{n+1}. 2%

(iv) Specify the argument of f_{n+1} in the recurrence formula. 2%

(b) The following table, Table 2 gives the minimum costs for $n = 4$. (Entries with asterisks ($*$) are infeasible — explain why.)

Derive the numbers in the $f_4(s)$ and the x_4^* column for $s = 10$, $s = 15$, $s = 20$. and $s = 25$.. 4%

Important: Explain your calculations clearly.

$D_4 = 25$ n=4	$P_4 = 15$	Prod cost $c_4 =$ €1400
s	$f_4(s)$	x_4^*
0	$*$	$*$
5	$*$	$*$
10	21200	15
15	14300	10
20	7400	5
25	500	0
30	600	0
35	700	0
40	800	0
45	900	0
50	1000	0
55	1100	0

Table 2: Month 4

(c) The following table, Table 3 (Month 3) gives the minimum costs for $n = 3$.

(i) Again, entries with asterisks ($*$) are infeasible. Explain why the case $s = 10$, $x_3 = 20$ is infeasible.. 2%

(ii) Derive the "non-asterisk" numbers in the $s = 5$ and $s = 10$ rows. 4%

Important: Explain your calculations clearly.

$D_3 = 30$	$P_3 = 35$	Prod cost $c_3 =$ €1100								
n=3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3		
s	0	5	10	15	20	25	30	35	$f_3(s)$	x_3^*
0	*	*	*	*	*	*	*	*	*	*
5	*	*	*	*	*	*	*	59800	59800	35
10	*	*	*	*	*	*	54400	53000	53000	35
15	*	*	*	*	*	49000	47600	46200	46200	35
20	*	*	*	*	43600	42200	40800	39400	39400	35
25	*	*	*	38200	36800	35400	34000	39600	34000	30
30	*	*	54800	31400	30000	28600	34200	39800	28600	25
35	*	27400	48000	24600	23200	28800	34400	40000	23200	20
40	22000	20600	41200	17800	23400	29000	34600	40200	17800	15
45	15200	13800	34400	18000	23600	29200	34800	40400	13800	5
50	8400	7000	34600	18200	23800	29400	35000	40600	7000	5

Table 3: Month 3

The following table, Table 4 (Month 2) is for use in the next part of Q.3

$D_2 = 20$	$P_2 = 40$	Prod cost $c_2 =$€1300									
$n = 2$	x_2										
s	0	5	10	15	20	25	30	35	40	$f_2(s)$	x_2^*
0	*	*	*	*	*	92300	92000	91700	91400	91400	40
5	*	*	*	*	92400	85600	85300	85000	86100	85000	35
10	*	*	*	86000	85700	78900	78600	79700	80800	78600	30
15	*	*	73100	79300	79000	72200	73300	74400	75500	72200	25
20	*	66700	66400	72600	72300	66900	68000	69100	70200	66400	10
25	60300	60000	59700	65900	67000	61600	62700	63800	66300	59700	10
30	53600	53300	53000	60600	61700	56300	57400	59900	59600	53000	5
35	46900	46600	47700	55300	56400	51000	53500	53200	52700	46600	5

Table 4: Month 2

(d) The following table, Table 5 (Month 1) gives the minimum costs for $n = 1$.

(i) Explain why $x_1 = 0$, $x_1 = 5$ and $x_1 = 10$ are infeasible. 2%

(ii) Derive the "non-asterisk" numbers in the $s = 0$ row. 4%
Important: Explain your calculations clearly.

$D_1 = 15$	$P_1 = 30$	Prod cost $c_1 =$ €1400							
$n = 1$	x_1								
s	0	5	10	15	20	25	30	$f_1(s)$	x_1^*
0	*	*	*	112400	113000	113600	114200	112400	15

Table 5: Month 1

(iii) From Table 5 (Month 1), find the minimum cost of meeting demand. 1%

(iv) Finally, what is the optimal production plan (optimal values for x_1, x_2, x_3 and x_4? 2%

4 An American put option (right to sell a share) has a strike price of €100 with 3 periods left to maturity. The current share price is €105. Analysis of the market indicates that the share price volatility per period is described by: the stock will go up by €2 with probability 0.2, stay the same with probability 0.3, decrease by €2 with probability 0.3 or decrease by €4 with probability 0.2.

Use a Dynamic Programming formulation to investigate whether or not to exercise the option, and if it is to be exercised when to exercise it. 25 %

5 (a) Define what is meant by a minimax strategy in a 2-player game. 3 %

(b) For the following matrix game

	A	B
A	(5,3)	(4,4)
B	(2,6)	(4,4)

Find the minimax strategies and payoffs for each player. If both players play their minimax strategies, what is the value of the game ? 10 %

(c) In the context of a 2-person game, define what a *Nash* equilibrium is. 3 %

(d) Does the game of part (b) have *Nash* equilibria ? Justify your answer. 9 %

6 Consider the (symmetric) *Cournot* duopoly game: Firm i, $i = 1, 2$ produces x_i items at a cost of

$$C(x_i) = \frac{x_i^2}{1000} + 3x_i + 20$$

. The items sell at a price of

$$p(x_1, x_2) = 5 - \frac{x_1 + x_2}{500}$$

each.

(a) Find the equilibrium of this game, and prove that it is a *Nash* equilibrium. 8 %

(b) Investigate this game if a collusive strategy is used. Contrast its solution with that of part (a). 8 %

(c) If the game is to be played repeatedly, does it ever pay to defect from the collusive strategy? In particular, consider the *stern strategy*: a firm produces the collusive number of items until the other firm defects, after which it reverts to producing the *Cournot* number of items. Using the discount factor ω per period, when is this stern strategy a *Nash* equilibrium ? 9 %

Algorithm 1 (Simplex Method)

begin (Start with a Canonical tableau s.t. $\mathbf{b} \geq \mathbf{0}$.)
  while NOT finished do
    if $c_j \geq 0$ for all j
      then **STOP** (Tableau is optimal.)
      else Select j s.t. $c_j < 0$.
    fi
    if $a_{ij} \leq 0$ for all $i = 1, \dots, m$
      then **STOP** (Problem is unbounded.)
    fi
    Select k such that:
    $\frac{b_k}{a_{kj}} = \min_i \left\{ \frac{b_i}{a_{ij}} \text{such that } a_{ij} > 0 \right\}$ (k attains the min.)
    Pivot on a_{kj}. (Divide Row k across by a_{kj} and add
  end … multiples of Row k to the rows above & below
end … … … introducing zeros into column j.)

Algorithm 2 (Dual Simplex Method)

begin (Start with a tableau s.t. $\mathbf{c} \geq \mathbf{0}$.)
  while NOT finished do
    if $b_i \geq 0$ for all i
      then **STOP** (Tableau is optimal.)
      else Select i s.t. $b_i < 0$.
    fi
    if $-a_{ij} \leq 0$ for all $j = 1, \dots, n$
      then **STOP** (Dual unbounded $\equiv$ Primal infeasible.)
    fi
    Select k such that:
    $\frac{c_k}{a_{ik}} = \max_j \left\{ \frac{c_j}{a_{ij}} \text{such that } a_{ij} < 0 \right\}$ (k attains max.)
    Pivot on a_{ik}. (Divide Row k across by a_{ik} and add
  end … multiples of Row k to the rows above & below
end … … … introducing zeros into column i.)