MS4018 Problem Sheet 6-2

1. For each of the following 1-d maps, calculate the fixed point(s) as a function of the parameter a, show that bifurcations occurs at critical values of $a = a_{c_i}$, determine the type of bifurcation, and sketch the bifurcation diagram x_e vs a.

   (a) $x' = 0.8x + ax(1 - \frac{x}{100})$

   (b) $x' = -a + x + e^{-x}$

   (c) $x' = \frac{ax}{1+bx}$

   (d) $x' = a + x - \frac{1}{2}x^2$

   (e) $x' = ax - \frac{1}{3}x^3$

2. The macroeconomic model

$$K' = (1-\delta)K + sY$$

describes how the capital (K) available for investment at the next time period consists of the depreciated capital currently available ($0 < \delta < 1$ is the rate of depreciation) plus savings - a proportion ($0 < s < 1$) of the production (Y) in the current period.

A commonly used class of production function express production as a function of the two factors capital and labour (L), i.e. $Y = F(K, L)$. It is called a "constant returns to scale" production function if increasing each factor by the same percentage amount increases production by a similar percentage amount. Mathematically this is achieved if "F is homogeneous of degree 1", i.e. $F(\beta K, \beta L) = \beta F(K, L)$ for any constant β.

We shall further assume that the labour force grows at a constant rate, i.e.

$$L' = (1+n)L$$

   (a) By defining the per capita quantities $k = K/L$ and $y = Y/L$, and assuming a constant returns to scale production function, show that the macroeconomic model may be rewritten as

$$k' = \frac{1-\delta}{1+n}k + \frac{s}{1+n}f(k)$$

   where $f(k) = F(k, 1)$

   (b) Consider the *Cobb-Douglas* constant returns to scale production function

$$F(K, L) = AK^{\alpha}L^{1-\alpha}$$

   where $A > 0$ is a technology dependent factor and $0 < \alpha < 1$ is the output elasticity of capital. Show that F is homogeneous of degree 1. With this production function, show that the macroeconomic model becomes

$$k' = \frac{1-\delta}{1+n}k + \frac{As}{1+n}k^{\alpha}$$

   With $\delta = 5\%$, $n = 1\%$, $\alpha = 0.5$ and $A = 1$, find the fixed points of the map and investigate their stability properties as a function of the parameter s. Are there any bifurcations? What would now happen if A is increased (i.e. using a more effective technology)?

(c) With the production function

$$F(K, L) = \frac{KL}{0.5K + 0.25L}$$

redo the analysis of part (b)

3. For the system

$$x' = y - x, \qquad y' = a - x^2$$

Find the fixed points as functions of the parameter a, and classify the bifurcations that occur as a varies.