MS4018 Problem Sheet 5

1. For each of the following, find their fixed points and calculate the index.

   (a) $\dot{x} = -x - 3y, \qquad \dot{y} = -x - 2y$

   (b) $\dot{x} = x + 2y, \qquad \dot{y} = -x + 3y$

   (c) $\dot{x} = x^2, \qquad \dot{y} = y$

   (d) $\dot{x} = xy, \qquad \dot{y} = x + y$

2. Use Index Theory to investigate the possibility of closed orbits for the following systems

   (a) $\dot{x} = xe^{-x}, \qquad \dot{y} = 1 + x^2 + y^2$

   (b) $\dot{x} = x(4 - x - 3y), \qquad \dot{y} = y(2 - x - y)$

   (c) $\dot{x} = y, \qquad \dot{y} = -x - y + x^2 + y^2$

3. Consider the system

$$\dot{x} = y, \qquad \dot{y} = -x - y + x^2 + y^2$$

   Use *Dulac*'s Criterion to show that the system has no closed orbits anywhere. What can be shown if

   (a) $g(x, y) = 1$

   (b) $g(x, y) = e^{\alpha x}$

4. Consider the system

$$\dot{x} = y + x(1 - \frac{1}{4} - x^2 - y^2), \qquad \dot{y} = -x + y(1 - x^2 - y^2)$$

   (a) Rewrite the system in polar coordinates.

   (b) Show that $\frac{\sqrt{3}}{2} - \epsilon < r \leq 1 + \epsilon$ is a trapping region for sufficiently small $\epsilon$.

   (c) Use the *Poincaré-Bendixson* Theorem to prove that there is a limit cycle.

   (d) What is the period ($T$) of the limit cycle ?
   Note :

$$T = \oint dt = \oint \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}}$$

5. For the flow of Question 4, choose a suitable surface of section, compute the resultant *Poincaré* map, find its fixed point(s) and check its/their stability properties. Use the P-map *Maple* code.