1. Find the fixed points of the flow

$$\dot{x} = -x + y + \frac{1}{2}xy, \qquad \dot{y} = -y + \frac{9}{8}x^2$$

and classify them.

   (a) Show that $V(\mathbf{x}) = 2x^2 + 2xy + 3y^2$ is a *Lyapunov* function for $\mathbf{0}$, and use it to estimate the basin of attraction.

   (b) Use Maple to compute the invariant manifolds of the saddle fixed points.

   (c) Does the solution of (b) help in re-evaluating the basin of attraction obtained in (a)?

2. Find a quadratic *Lyapunov* function for the system:

$$\dot{x} = y - x^3, \qquad \dot{y} = -x - y^3$$

3. Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{0} = \mathbf{f}(\mathbf{x}_e)$ for which a global strict *Lyapunov* function exists. Prove that the system can have no periodic orbits.

4. Use the *Lyapunov* equation to determine whether the following linear flows are stable.

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\mathbf{x}, \qquad \dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -1 & -4 \end{pmatrix}\mathbf{x}$$

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix}\mathbf{x}$$

5. Consider the linear map $\mathbf{x}' = A\mathbf{x}$. Derive the "Discrete *Lyapunov* Equation" satisfied by a quadratic *Lyapunov* function and state the theorem that an asymptotically stable system must satisfy when such a function is used. Hence determine whether the following map is stable:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix}\mathbf{x}$$