MS4018 Problem Sheet 4-1

1. Show that the origin is a saddle for the system:

$$\dot{x} = y, \qquad \dot{y} = 3x - 2y + y^2$$

Compute its stable and unstable manifolds using Maple; in this example they are both curves.
Note: *In principle*, solving the differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad y(x_e) = y_e$$

generates an orbit "going through" the fixed point (x_e, y_e), and hence both manifolds satisfy this equation. How do you tweak the integration method to pick the desired curve ?

2. For the origin of the flow of Question 1,

   (a) find the invariant manifolds associated with the linearised system.

   (b) Compute the power series representation of the *unstable* manifold.

   (c) Find the state transformation which aligns the new axes with the manifolds of the linearised system, and then compute the power series representation of the *stable* manifold.

3. Define the stable and unstable manifolds of a saddle point for a 2-d invertible map.

   Do these change if the map is non-invertible?

   Unlike the case for flows, invariant manifolds of maps are not orbits of the map. What can be said about any point on these manifolds ?

4. (a) Show that the origin of the map

$$x' = y, \qquad y' = \frac{3}{4}x + \frac{1}{4}x^2 - \frac{1}{6}y^2 + y$$

   is a saddle.

   (b) Prove that if its stable manifold can be written as $y = h(x)$, then it satisfies

$$\frac{3}{4}x + \frac{1}{4}x^2 - \frac{1}{6}(h(x))^2 + h(x) = h(h(x)) \qquad (1)$$

   (c) Hence show that

$$h(x) = -\frac{1}{2}x - \frac{1}{6}x^2, \qquad -9 < x < 6 \qquad (2)$$

   (d) Show that the unstable manifold satisfies the same functional equation as the stable manifold (Eq (1)). So, how do you know that Eq(2) is actually the stable manifold?

5. For the map of Question 4, find the power series approximation of the unstable manifold of the origin.

6. Show that the map $x' = 3x(1-x)$ has a non-hyperbolic fixed point at $x_e = 2/3$. By changing variable to $z = x - 2/3$, investigate the stability of the fixed point $z_e = 0$ by considering the first and/or second iterates of the resulting map.

7. Show that the map

$$x' = 3x\left(1 - x - \frac{1}{2}y\right), \qquad y' = \frac{7}{2}y\left(1 - y - \frac{9}{8}x\right)$$

has a non-hyperbolic fixed point. Use Centre Manifold Theory to investigate the stability of this fixed point.